

RIGID GEOMETRIC STRUCTURES, ISOMETRIC ACTIONS, AND ALGEBRAIC QUOTIENTS

JINPENG AN

ABSTRACT. By using a Borel density theorem for algebraic quotients, we prove a theorem concerning isometric actions of a Lie group G on a smooth or analytic manifold M with a rigid A-structure σ . It generalizes Gromov's centralizer and representation theorems to the case where $R(G)$ is split solvable and $G/R(G)$ has no compact factors, strengthens a special case of Gromov's open dense orbit theorem, and implies that for smooth M and simple G , if Gromov's representation theorem does not hold, then the local Killing fields on \widetilde{M} are highly non-extendable. As applications of the generalized centralizer and representation theorems, we prove (1) a structural property of $\text{Iso}(M)$ for simply connected compact analytic M with unimodular σ , (2) three results illustrating the phenomenon that if G is split solvable and large then $\pi_1(M)$ is also large, and (3) two fixed point theorems for split solvable G and compact analytic M with non-unimodular σ .

0. NOTATION AND CONVENTIONS

Throughout this paper, M denotes a connected C^ε manifold, where $\varepsilon = \infty$ or ω (here C^ω means real analytic). In most cases, M is endowed with a rigid C^ε geometric structure of algebraic type (abbreviated as A-structure) in the sense of Gromov [21]. We denote the universal cover of M by \widetilde{M} , with covering map $\pi : \widetilde{M} \rightarrow M$. For simplicity, sometimes we also denote $\Gamma = \pi_1(M)$. If f is a local C^ε diffeomorphism of M defined around $x \in M$, we denote the germ of f at x by $\langle f \rangle_x$. Similarly, if v is a C^ε local vector field defined around x , we denote the germ of v at x by $\langle v \rangle_x$.

Suppose that M is endowed with a C^ε geometric structure σ . We denote by $\text{Iso}(M)$ the C^ε isometry group of M , by $\text{Iso}^{\text{germ}}(M)$ the groupoid of germs of local C^ε isometries of M , by $\text{Kill}(M)$ the Lie algebra of C^ε Killing fields on M , and by $\text{Kill}_x^{\text{germ}}(M)$ the Lie algebra of germs at x of local C^ε Killing fields defined around x . We also denote the evaluation map at x from $\text{Kill}(M)$ or $\text{Kill}_x^{\text{germ}}(M)$ to $T_x M$ by the same symbol ev_x . Note that if σ is rigid, then $\text{Iso}(M)$ is a Lie group, $\text{Kill}(M)$ and $\text{Kill}_x^{\text{germ}}(M)$ are finite-dimensional.

Remark 0.1. We prefer to work with the groupoid $\text{Iso}^{\text{germ}}(M)$ rather than the more commonly used pseudogroup $\text{Iso}^{\text{loc}}(M)$ of local isometries. This makes the notion of Zariski hulls, which will be introduced later, easier to define and use. The orbits of $\text{Iso}^{\text{germ}}(M)$ and $\text{Iso}^{\text{loc}}(M)$ are the same. For basic facts concerning groupoids, the reader may refer to [26].

We denote by \mathcal{L} the operation which takes a Lie group to its Lie algebra. If a Lie group is denoted by a capital Latin letter, we also denote its Lie algebra by the

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corresponding small Gothic letter. If a group G acts on a set X , we denote the set of G -orbits in X by X/G , denote the set of G -fixed points in X by X^G , and denote the stabilizer of $x \in X$ in G by $G(x)$. If G is a Lie group and $G(x)$ is closed, we denote $\mathfrak{g}(x) = \mathcal{L}(G(x))$.

Whenever we say a Lie group G acts on M by C^ε isometries, we always assume that the action map $G \times M \rightarrow M$ is C^ε . In this case, for every $x \in M$, the surjective linear map $\iota_x : \mathfrak{g} \rightarrow T_x Gx$ defined by $\iota_x(X) = \frac{d}{dt}|_{t=0} \exp(-tX)x$ has kernel $\mathfrak{g}(x)$. We refer to the map $\iota : \mathfrak{g} \rightarrow \text{Kill}(M)$, $\text{ev}_x(\iota(X)) = \iota_x(X)$ as the induced infinitesimal action of \mathfrak{g} on M . It is well-known that ι is a Lie algebra homomorphism and is G -equivariant with respect to the adjoint representation and the natural representation $g \mapsto dg$ of G in $\text{Kill}(M)$. We refer to $\mathcal{G} = \iota(\mathfrak{g})$ as the Lie algebra of Killing fields induced by the G -action. If G is connected with universal cover \tilde{G} , then the G -action induces a C^ε isometric action of \tilde{G} on \tilde{M} . We denote by $\tilde{\mathcal{G}}$ the Lie algebra of Killing fields induced by the \tilde{G} -action on \tilde{M} , which is obviously equal to the image of \mathcal{G} under the natural injection $\text{Kill}(M) \rightarrow \text{Kill}(\tilde{M})$.

Let \mathbb{k} be a locally compact non-discrete field. By a \mathbb{k} -variety (resp. \mathbb{k} -group) we mean an algebraic variety (resp. linear algebraic group) defined over \mathbb{k} . We use a subscript “0” to denote the identity component of a \mathbb{k} -group (resp. Lie group) under the Zariski (resp. Hausdorff) topology. The set (resp. group) of \mathbb{k} -rational points of a \mathbb{k} -variety (resp. \mathbb{k} -group) is indicated by a subscript “ \mathbb{k} ”. Unless otherwise specified, whenever a topological statement concerning such a set (resp. group) is made, the topology is understood as the Hausdorff one.

We refer to the set (resp. group) of \mathbb{R} -rational points of an \mathbb{R} -variety (resp. \mathbb{R} -group) as a real algebraic variety (resp. group). A real algebraic group can be also recognized as an algebraic subgroup of $\text{GL}(\mathcal{V})$, where \mathcal{V} is some finite-dimensional real vector space. If G is a subgroup of $\text{GL}(\mathcal{V})$, we denote by \overline{G} the Zariski closure of G in $\text{GL}(\mathcal{V})$. Note that if G is a real algebraic group, G_0 refers to the identity component of G under the Hausdorff topology.

Let G be a Lie group, and let ρ be a finite-dimensional real representation of G . Motivated by [32] (and [2, 25]), we say that G is *discompact with respect to* ρ (or ρ -*discompact* for short) if $\overline{\rho(G)}$ has no proper normal cocompact algebraic subgroups. We make the convention that G is ρ -discompact if the representation space is 0. Following [22], if G is connected and solvable, we say that G is *split solvable* if all eigenvalues of $\text{Ad}(g)$ are real for every $g \in G$.

Let \mathfrak{a} be a (real) Lie algebra. We write $\mathfrak{a}_1 < \mathfrak{a}$ (resp. $\mathfrak{a}_1 \triangleleft \mathfrak{a}$) to indicate that \mathfrak{a}_1 is a subalgebra (resp. ideal) of \mathfrak{a} . Recall that a Lie algebra \mathfrak{b} is called a subquotient of \mathfrak{a} if there exist $\mathfrak{a}_1 < \mathfrak{a}$ and $\mathfrak{a}_2 \triangleleft \mathfrak{a}_1$ such that $\mathfrak{b} \cong \mathfrak{a}_1/\mathfrak{a}_2$. We denote this relation by $\mathfrak{b} \prec \mathfrak{a}$. From the Levi decomposition, it is easy to see that if \mathfrak{b} is semisimple and $\mathfrak{b} \prec \mathfrak{a}$, then \mathfrak{a} has a subalgebra isomorphic to \mathfrak{b} .

1. INTRODUCTION

1.1. Background. Motivated by the work of Zimmer [39] on rigidity of group actions (later called the Zimmer program), Gromov [21] introduced the notion of rigid geometric structures, and proved several deep results which have had profound influence on the geometry and dynamics of group actions (see [1, 4, 5, 18, 24, 41, 42] and the references therein). For example, for a connected C^ε manifold M with a

rigid C^ε A-structure, where $\varepsilon = \infty$ or ω , Gromov proved the following fundamental theorems.

- *Gromov's open dense orbit theorem:* If $\text{Iso}^{\text{germ}}(M)$ has a dense orbit in M , then it has an open dense orbit.
- *Gromov's centralizer theorem:* Suppose that $\varepsilon = \omega$, M is compact, and a connected noncompact simple Lie group G acts faithfully, analytically, and isometrically on M and preserves a finite smooth measure on M . Let \mathcal{Z} be the centralizer of \tilde{G} in $\text{Kill}(\tilde{M})$. Then for a.e. $\tilde{x} \in \tilde{M}$, we have $T_{\tilde{x}}\tilde{G}\tilde{x} \subset \text{ev}_{\tilde{x}}(\mathcal{Z})$.
- *Gromov's representation theorem:* Under the conditions of Gromov's centralizer theorem, if $\rho : \pi_1(M) \rightarrow \text{GL}(\mathcal{Z})$ is the representation induced by the deck transformations, then $\overline{\rho(\pi_1(M))}$ has a Lie subgroup locally isomorphic to G .

Note that rigid geometric structures generalize Cartan's structures of finite type, and rigid A-structures include pseudo-Riemannian structures, linear connections, pseudo-Riemannian conformal structures in dimension ≥ 3 , etc. The reader may refer to [12, 17, 18, 24, 41, 42] for surveys of Gromov's theory. For detailed discussions and further developments of the above theorems, see, e.g., [2, 3, 6, 9, 10, 14, 16, 19, 25, 28, 36, 37].

1.2. The general theorem. The first goal of this paper is to prove a general theorem which, in a certain sense, unifies and generalizes the above three theorems of Gromov. To state the result, we need to recall and introduce more notation and terminology. Let M be a connected C^ε manifold with a rigid C^ε A-structure σ , where $\varepsilon = \infty$ or ω . It is proved in [21] (see also [3, 16]) that there exists an open dense subset $M_{\text{reg}} \subset M$ invariant under $\text{Iso}^{\text{germ}}(M)$ such that for some sufficiently large integer k , every $x \in M_{\text{reg}}$ has a neighborhood U_x such that any infinitesimal isometry of order k sending x to $y \in U_x$ extends to a local C^ε isometry. Moreover, if $\varepsilon = \omega$ and M is compact, we may take $M_{\text{reg}} = M$. Throughout this paper, we fix M_{reg} for a given (M, σ) , and set $M_{\text{reg}} = M$ if $\varepsilon = \omega$ and M is compact.

Let a connected Lie group G act on M by C^ε isometries. For a finite Borel measure μ on M , we denote $G_\mu = \{g \in G \mid g_*(\mu) = \mu\}$, which is a closed subgroup of G (see, e.g., [30, Prop. 1.1]). We say a subset $M' \subset M$ is G -saturated if $x \in M'$, $y \in M$, and $\overline{Gx} = \overline{Gy}$ imply that $y \in M'$. The interest of this notion lies in the fact that if a G -minimal set $M_0 \subset M$ intersects a G -saturated set M' , then $M_0 \subset M'$.

Let \mathcal{V} be a subspace of $\text{Kill}(M)$. We define various types of centralizers of \mathcal{V} as follows.

- We say that $\varphi \in \text{Iso}^{\text{germ}}(M)$ centralizes \mathcal{V} if $d\varphi(\langle v \rangle_x) = \langle v \rangle_y$ for all $v \in \mathcal{V}$, where x and y are the source and target of φ , respectively. We refer to the subgroupoid of $\text{Iso}^{\text{germ}}(M)$ consisting of all germs centralizing \mathcal{V} as the *centralizer of \mathcal{V} in $\text{Iso}^{\text{germ}}(M)$* , and denote it by $\text{Iso}^{\text{germ}}(M)^\mathcal{V}$.
- For $x \in M$, we refer to the centralizer in $\text{Kill}_x^{\text{germ}}(M)$ of the image of \mathcal{V} under the natural injection $\text{Kill}(M) \rightarrow \text{Kill}_x^{\text{germ}}(M)$ (see Theorem 3.1(3) below) as the *centralizer of \mathcal{V} in $\text{Kill}_x^{\text{germ}}(M)$* , and denote it by $\text{Kill}_x^{\text{germ}}(M)^\mathcal{V}$.
- We refer to the centralizer in $\text{Kill}(\tilde{M})$ of the image of \mathcal{V} under the natural injection $\text{Kill}(M) \rightarrow \text{Kill}(\tilde{M})$ as the *centralizer of \mathcal{V} in $\text{Kill}(\tilde{M})$* , and denote it by $\text{Kill}(\tilde{M})^\mathcal{V}$.

Note that the space \mathcal{Z} in Gromov's centralizer and representation theorems coincides with $\text{Kill}(\widetilde{M})^{\mathcal{G}}$.

Now we assume that \mathcal{V} is G -invariant under the natural representation of G in $\text{Kill}(M)$. Then we have a representation $\rho_{\mathcal{V}} : G \rightarrow \text{GL}(\mathcal{V})$, $g \mapsto dg|_{\mathcal{V}}$. We are interested in the $\rho_{\mathcal{V}}$ -discompactness of G . For the $\mathcal{V} \supset \mathcal{G}$ case, we have the following criterion.

Lemma 1.1. *Let M and G be as above, and let \mathcal{V} be a G -invariant subspace of $\text{Kill}(M)$ containing \mathcal{G} .*

- (1) *If the radical $R(G)$ of G is split solvable, the semisimple group $G/R(G)$ has no compact factors, and there exists a sequence $\mathcal{G} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_k = \mathcal{V}$ of $R(G)$ -invariant subspaces of \mathcal{V} with $\dim \mathcal{V}_i / \mathcal{V}_{i-1} = 1$, then G is $\rho_{\mathcal{V}}$ -discompact.*
- (2) *If the G -action on M is faithful, then the converse of (1) also holds.*

Consider the natural representation of $\Gamma = \pi_1(M)$ in $\text{Kill}(\widetilde{M})$ induced by the deck transformations. Let \mathcal{W} be a Γ -invariant subspace of $\text{Kill}(\widetilde{M})$. For a subgroupoid \mathcal{A} of $\text{Iso}^{\text{germ}}(M)$ and a subalgebra \mathfrak{a}_x of $\text{Kill}_x^{\text{germ}}(M)$, we will define the Zariski hulls of Γ in \mathcal{A} and \mathfrak{a}_x relative to \mathcal{W} , denoted by $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A})$ and $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x)$ respectively, which encode some information of Γ . Here we only note that $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A})$ is a subgroupoid of \mathcal{A} , $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x)$ is a subalgebra of \mathfrak{a}_x , and if $\mathcal{W} = 0$, then $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}) = \mathcal{A}$ and $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x) = \mathfrak{a}_x$.

Our general theorem is as follows.

Theorem 1.2. *Let M and G be as above, \mathcal{V} be a G -invariant subspace of $\text{Kill}(M)$, \mathcal{W} be a Γ -invariant subspace of $\text{Kill}(\widetilde{M})^{\mathcal{G}}$, and μ be a finite Borel measure on M . Suppose that G is $\rho_{\mathcal{V}}$ -discompact and $\ker(\rho_{\mathcal{V}}) \cdot G_{\mu} = G$. Then there exists a G -saturated constructible set $M' \subset M_{\text{reg}}$ with $\mu(M_{\text{reg}} \setminus M') = 0$ such that for every $x \in M'$, we have*

- (1) *the $\text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Iso}^{\text{germ}}(M)^{\mathcal{V}})$ -orbit of x contains a G -invariant open dense subset of \overline{Gx} , and*
- (2) *$T_x Gx \subset \text{ev}_x(\text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}}))$.*

1.3. The centralizer and representation theorems. The $\mathcal{W} = 0$ case of Theorem 1.2(1) can be viewed as a “global” version of Gromov's centralizer theorem, which asserts that the $\text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$ -orbit of x contains not only Gx but also an open dense subset of \overline{Gx} . Thus it generalizes Gromov's open dense orbit theorem (see Theorem 8.3 below). Theorem 1.2(2) is the “infinitesimal” version of Theorem 1.2(1) and will be derived from it. By taking $\mathcal{W} = 0$ in Theorem 1.2(2), one easily obtain the following Theorem 1.3(1), which generalizes Gromov's centralizer theorem. But Theorem 1.2(2) indeed implies that for any Γ -invariant subspace \mathcal{W} of $\text{Kill}(\widetilde{M})^{\mathcal{G}}$, the first assertion of Theorem 1.3(1) remains true if we replace $\text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}}$ by its subalgebra $\text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}})$. By taking $\mathcal{W} = \text{Kill}(\widetilde{M})^{\mathcal{V}}$, we obtain Theorem 1.3(2) in a very direct way, which generalizes Gromov's representation theorem.

Theorem 1.3. *Let M be a connected C^{ε} manifold with a rigid C^{ε} A -structure and a finite Borel measure μ , G be a connected Lie group which acts on M by C^{ε} isometries and preserves μ , and $\mathcal{V} \supset \mathcal{G}$ be a G -invariant subspace of $\text{Kill}(M)$ such that G is $\rho_{\mathcal{V}}$ -discompact. Then there exists a G -saturated constructible set $M' \subset$*

M_{reg} with $\mu(M_{\text{reg}} \setminus M') = 0$ such that for every $x \in M'$, the following assertions hold.

- (1) $T_x Gx \subset \text{ev}_x(\text{Kill}_x^{\text{germ}}(M)^\vee)$ and $\mathfrak{g}(x) \triangleleft \mathfrak{g}$. If moreover $\varepsilon = \omega$, then $T_{\tilde{x}} \tilde{G}\tilde{x} \subset \text{ev}_{\tilde{x}}(\text{Kill}(\tilde{M})^\vee)$ for every $\tilde{x} \in \pi^{-1}(x)$.
- (2) If $\varepsilon = \omega$ and $\rho : \pi_1(M) \rightarrow \text{GL}(\text{Kill}(\tilde{M})^\vee)$ is the representation induced by the deck transformations, then $\text{ad}(\mathfrak{g}/\mathfrak{g}(x)) \prec \mathcal{L}(\overline{\rho(\pi_1(M))})$.

Remark 1.1. (1) The case of Theorem 1.3 where G is semisimple without compact factors strengthens Gromov's centralizer and representation theorems. Indeed, Lemma 1.1 implies that G is ρ_V -discompact for $V = \text{Kill}(M)$. Since $\text{Kill}(\tilde{M})^{\text{Kill}(M)} \subset \text{Kill}(\tilde{M})^\mathcal{G}$, the last assertion of Theorem 1.3(1) strengthens Gromov's centralizer theorem. On the other hand, as is well-known, if μ is smooth and the G -action is faithful, then the G -action is a.e. locally free (see Lemma 8.2). In particular, there exists $x \in M'$ such that $\mathfrak{g}(x) = 0$. Hence Theorem 1.3(2) implies that $\overline{\rho(\pi_1(M))}$ has a Lie subgroup locally isomorphic to G , where the representation space of ρ is $\text{Kill}(\tilde{M})^{\text{Kill}(M)}$.

(2) The conditions of Theorem 1.3 can be also satisfied for $V = \text{Kill}(M)$ and non-semisimple G . Indeed, if the geometric structure is unimodular (e.g., a pseudo-Riemannian structure or a linear connection plus a volume density), then there exists a connected closed normal subgroup G of $\text{Iso}(M)$ such that $\text{Iso}(M)/G$ is locally isomorphic to a compact Lie group, and such that the conditions of Theorem 1.3 are satisfied for $V = \text{Kill}(M)$ (see Remark 9.1).

(3) If $R(G)$ is split solvable and $G/R(G)$ has no compact factors, then by Lemma 1.1, G is ρ_G -discompact. In this general case, the G -action may be not a.e. locally free. But Theorem 1.3(2) relates the size of $\overline{\rho(\pi_1(M))}$ to the sizes of the stabilizers. This is sufficient for our applications (Theorems 1.5 and 1.7 below).

(4) If G is split solvable, then it is amenable and ρ_G -discompact. Thus we can deduce from Theorem 1.3 the centralizer and representation theorems for split solvable G which dispenses with the finite invariant measure (see Corollary 9.2).

(5) We do not need μ to be smooth (or "Zariski" in the sense of [9]). But if μ is smooth, then M' contains a G -invariant open dense subset of M , due to the fact that a constructible subset of a topological space contains an open dense subset of its closure. (This is well-known for Noetherian spaces. We will prove it for general topological spaces in Lemma 2.1.)

(6) The proofs of Gromov's representation theorem and Theorem 1.3(2) are based on the fact that if $\varepsilon = \omega$, then every local Killing field on \tilde{M} can be extended to a global Killing field. By using Theorem 1.2(2), we can prove that if $\varepsilon = \infty$ and Gromov's representation theorem does not hold, then the local Killing fields on \tilde{M} must be highly non-extendable (see Theorem 8.1 below).

1.4. Applications of Theorem 1.3. Our first application of Theorem 1.3 is a structural property of $\text{Iso}(M)$ for a simply connected compact analytic M with a rigid unimodular A-structure. A theorem of D'Ambra [11] asserts that if M is a simply connected compact analytic Lorentz manifold then $\text{Iso}(M)$ is compact.

Although D'Ambra's theorem no longer holds for general pseudo-Riemannian structures (see [11, Sect. 5] for a counterexample), Gromov [21, Thm. 3.7.A] proved the following result, which we call it Gromov's compactness theorem.

- *Gromov's compactness theorem:* Let M be a connected and simply connected compact analytic manifold with a rigid unimodular analytic A-structure. Then $\text{Iso}(M)$ is a compact extension of a connected abelian group.

We will prove the following result, which imposes another restriction on the possible choices of $\text{Iso}(M)$ for M as in Gromov's compactness theorem (e.g., the special Euclidean group $\text{SO}(n) \ltimes \mathbb{R}^n$ can not serve as $\text{Iso}(M)_0$).

Theorem 1.4. *Let M be a connected and simply connected compact analytic manifold with a rigid unimodular analytic A-structure. If $\text{Iso}(M)_0$ is nontrivial, then it either is compact semisimple or has a non-discrete center.*

Remark 1.2. If the geometric structure is a pseudo-Riemannian structure or a linear connection plus a volume density, the compactness condition for M in Theorem 1.4 can be replaced by the condition that M is geodesically complete and has finite volume (see Remark 9.3).

The second application concerns the relationship between the structure of G and the fundamental group of M . Let M be a connected analytic manifold with a rigid unimodular analytic A-structure of finite volume, and let a connected Lie group G act faithfully, analytically, and isometrically on M . It is commonly believed that if G is “large”, then $\pi_1(M)$ must also be “large”. A typical example is Gromov's representation theorem, which implies that if G is simple, then $\pi_1(M)$ is non-amenable. For more results in this direction, see [18, 41, 42] and the references therein. By using Theorem 1.3, we will prove the following theorem, which provides three results in this spirit for split solvable G .

Theorem 1.5. *Let M be a connected analytic manifold with a rigid unimodular analytic A-structure of finite volume, and let a connected split solvable Lie group G act faithfully, analytically, and isometrically on M .*

- (1) *If G is non-abelian, then $\pi_1(M)$ is infinite.*
- (2) *If G is not at most 2-step nilpotent, then $\pi_1(M)$ is not virtually abelian.*
- (3) *If G is non-nilpotent, then $\pi_1(M)$ is not virtually nilpotent.*

Remark 1.3. (1) The case of Theorem 1.5(1) where M is compact can be also deduced from Gromov's compactness theorem.

- (2) Recall that a group is *virtually abelian* (resp. *virtually nilpotent*) if it has an abelian (resp. nilpotent) subgroup of finite index. By Gromov's polynomial growth theorem [20], a finitely generated group (e.g., $\pi_1(M)$ for a compact M) is virtually nilpotent if and only if it has polynomial growth.

We also use Theorem 1.3 to prove two fixed point theorems for isometric actions of split solvable Lie groups. As is well-known, solvable group actions tend to have fixed points. For example, continuous affine actions of a solvable lcsc group (which is amenable) on a compact convex subset of a locally convex topological vector space have fixed points (see [38]). As another example, the Borel fixed point theorem [8] asserts that \mathbb{k} -regular actions of a connected \mathbb{k} -split solvable \mathbb{k} -group on a complete \mathbb{k} -variety \mathbf{V} with $\mathbf{V}_{\mathbb{k}} \neq \emptyset$ have fixed points in $\mathbf{V}_{\mathbb{k}}$. For isometric actions of a split solvable Lie group, we will prove the following result.

Theorem 1.6. *Let M be a connected and simply connected compact analytic manifold with a rigid non-unimodular analytic A-structure, and let G be a connected split solvable Lie subgroup of $\text{Iso}(M)$. Suppose that*

- (1) *G is discrepant with respect to the restriction to G of the adjoint representation of $\text{Iso}(M)$, and*
- (2) *$\text{Iso}(M)_0$ has a discrete center.*

Then any G -minimal set in M consists of a single point. In particular, G has a fixed point in M .

Under much weaker conditions, especially without any assumption on $\text{Iso}(M)_0$, we can prove that the commutator group (G, G) always has a fixed point.

Theorem 1.7. *Let M be a connected compact analytic manifold with a rigid non-unimodular analytic A-structure, and let G be a connected split solvable Lie group which acts analytically and isometrically on M . Suppose that either*

- (1) *$\pi_1(M)$ is finite, or*
- (2) *$\pi_1(M)$ is virtually nilpotent and $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, \mathfrak{g}]$.*

Then (G, G) acts trivially on any G -minimal set in M . In particular, (G, G) has a fixed point in M .

- Remark 1.4.*
- (1) Condition (1) in Theorem 1.6 is satisfied if H is a connected noncompact semisimple Lie subgroup of $\text{Iso}(M)$ with an Iwasawa decomposition $H = KAN$ and $G = AN$ (see Remark 9.4).
 - (2) Many important split solvable Lie groups G satisfy the condition $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, \mathfrak{g}]$. The simplest example is the “ $ax + b$ ” group. Another typical class of examples is $G = AN$, where KAN is an Iwasawa decomposition of a noncompact semisimple Lie group (in this case we have $(G, G) = N$). Note that if G satisfies the condition, then so does any quotient group of G .
 - (3) The non-unimodularity of the geometric structures in Theorems 1.6 and 1.7 will not be used in the proofs. But Theorems 1.4 and 1.5 imply that Theorems 1.6 and 1.7 are meaningful only for non-unimodular structures.

1.5. The main idea in the proof and algebraic quotients. As is well-known, the notion of algebraic hulls [38] is widely used in the study of Gromov’s centralizer theorem and related topics. However, it seems to the author that the algebraic hull method is not sufficient to prove Theorem 1.2 in full generality. Our proof of Theorem 1.2 uses some idea in Gromov’s proof of his centralizer theorem outlined in [21, Sec. 5.2]. But there are essential improvements. We construct a continuous geometric structure by augmenting σ with the geometric structures associated with the jets of Killing fields in \mathcal{V} and \mathcal{W} . The geometric structure associated with \mathcal{W} has its range in an algebraic quotient endowed with the quotient topology, and can be only expected to be continuous. But it has the advantage that it relates directly to $\pi_1(M)$. Then we consider the Gauss map of the augmented geometric structure, which also has range in an algebraic quotient and is continuous. Although the quotient topology on the algebraic quotient is in general not Hausdorff, the continuity of the Gauss map enables us to investigate interesting topological properties.

A crucial tool in the proof of Theorem 1.2 is a Borel density theorem for algebraic quotients. Let \mathbb{k} be a locally compact non-discrete field, and let \mathbf{G} be a \mathbb{k} -group. Following [32], we say that \mathbf{G} is \mathbb{k} -discompact if any \mathbb{k} -homomorphism $\mathbf{G} \rightarrow \mathbf{H}$ of

\mathbb{k} -groups with $\mathbf{H}_{\mathbb{k}}$ compact is trivial. One version of the Borel density theorem is as follows (see [32]).

- *Borel density theorem:* If a \mathbb{k} -discompact \mathbb{k} -group \mathbf{G} acts \mathbb{k} -regularly on a \mathbb{k} -variety \mathbf{V} with $\mathbf{V}_{\mathbb{k}} \neq \emptyset$ and $\mathbf{G}_{\mathbb{k}}$ preserves a finite Borel measure μ on $\mathbf{V}_{\mathbb{k}}$, then μ -a.e. point in $\mathbf{V}_{\mathbb{k}}$ is fixed by \mathbf{G} .

Now we assume that $\text{char } \mathbb{k} = 0$. Let \mathbf{G} and \mathbf{H} be \mathbb{k} -groups, and let \mathbf{V} be a \mathbb{k} -variety with $\mathbf{V}_{\mathbb{k}} \neq \emptyset$. Suppose that $\mathbf{G} \times \mathbf{H}$ acts \mathbb{k} -regularly on \mathbf{V} . We endow the algebraic quotient $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ with the quotient Borel structure. Then $\mathbf{G}_{\mathbb{k}}$ acts measurably on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$. We will prove and use the following result.

Theorem 1.8. *Let \mathbf{G} , \mathbf{H} , and \mathbf{V} be as above such that \mathbf{G} is \mathbb{k} -discompact, and let μ be a finite Borel measure on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ such that either*

- (1) μ is $\mathbf{G}_{\mathbb{k}}$ -invariant, or
- (2) $\mathbb{k} = \mathbb{R}$ and μ is invariant under some Zariski dense Lie subgroup of $\mathbf{G}_{\mathbb{R}}$.

Then for μ -a.e. $x \in \mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$, $\mathbf{G}_{\mathbb{k}}(x)$ is an open subgroup of finite index in $\mathbf{G}_{\mathbb{k}}$.

It is necessary to point out the following difference between Theorem 1.8 and the usual Borel density theorem, which already occurs for $\mathbb{k} = \mathbb{R}$: Although the $\mathbb{k} = \mathbb{R}$ (and μ is $\mathbf{G}_{\mathbb{R}}$ -invariant) case of Theorem 1.8 implies that the set of $(\mathbf{G}_{\mathbb{R}})_0$ -fixed points in $\mathbf{V}_{\mathbb{R}}/\mathbf{H}_{\mathbb{R}}$ is μ -conull, $\mathbf{G}_{\mathbb{R}}$ (which is Zariski connected by the discreteness) may have no fixed points in $\mathbf{V}_{\mathbb{R}}/\mathbf{H}_{\mathbb{R}}$ (see Remark 6.1).

1.6. Organization of the paper. In Section 2 we prove some auxiliary results concerning constructible sets, G -saturated sets, and subquotients of Lie algebras. In Section 3 we briefly review Gromov's theory of rigid geometric structures. In Section 4 we define Zariski hulls of fundamental groups and prove some of their basic properties. In Section 5 we study discreteness groups and prove Lemma 1.1. Then in Section 6 we prove Theorem 1.8. The proof is based on the usual Borel density theorem. Theorem 1.2 is proved in Section 7. We first prove Theorem 1.2(1) by using Theorem 1.8, and then prove Theorem 1.2(2) by passing from global to infinitesimal information. In Section 8 we provide the details of how Theorem 1.2 unifies Gromov's theorems, in particular, the proof of Theorem 1.3. Theorems 1.4–1.7 are proved in Section 9.

2. AUXILIARY RESULTS

In this section we prove some basic properties of constructible sets, G -saturated sets, and subquotients of Lie algebras. The reader may skip this section during the first reading and return to it later as reference.

2.1. Constructible sets. Let X be a topological space. Recall that a subset of X is *constructible* if it is a finite union of locally closed sets. It is well-known that if X is Noetherian, then every constructible subset of X contains an open dense subset of its closure ([8, AG.1.3]). We prove that this remains true for arbitrary X .

Lemma 2.1. *Let X be a topological space. If $Y \subset X$ is constructible, then it contains an open dense subset of \overline{Y} .*

Proof. Suppose $Y = \bigcup_{i=1}^k Y_i$, where each Y_i is locally closed. Denote $Z_i = \overline{Y_i} \setminus Y_i$, $Z = \bigcup_{i=1}^k Z_i$. We prove that $W = \overline{Y} \setminus Z$ is contained in Y and is open dense in \overline{Y} .

We first prove that $W \subset Y$. Since

$$Y \cup Z = \left(\bigcup_{i=1}^k Y_i \right) \cup \left(\bigcup_{i=1}^k Z_i \right) = \bigcup_{i=1}^k (Y_i \cup Z_i) = \bigcup_{i=1}^k \overline{Y_i} = \overline{\bigcup_{i=1}^k Y_i} = \overline{Y},$$

we have $W = \overline{Y} \setminus Z = (Y \cup Z) \setminus Z = Y \setminus Z \subset Y$. Now we show that W is open in \overline{Y} . Since Y_i is locally closed, it is open in $\overline{Y_i}$. So Z_i is closed in $\overline{Y_i}$, and hence is closed in X . Thus Z is closed, and hence W is open in \overline{Y} . Finally we prove that W is dense in \overline{Y} . If not, then Z contains a nonempty open subset of \overline{Y} . Let i_0 be the minimal index such that $\bigcup_{i=1}^{i_0} Z_i$ contains a nonempty open subset, say U , of \overline{Y} . We claim that $U \not\subset Z_i$ for any $1 \leq i \leq i_0$. Indeed, if $U \subset Z_i$, then U is a nonempty open subset of $\overline{Y_i}$ with $U \cap Y_i = \emptyset$, which is impossible. This in particular implies that $i_0 > 1$ and $U \not\subset Z_{i_0}$. So $\emptyset \neq U \setminus Z_{i_0} \subset \bigcup_{i=1}^{i_0-1} Z_i$. Since Z_{i_0} is closed, $U \setminus Z_{i_0}$ is open in \overline{Y} . This conflicts with the minimality of i_0 . Thus W is dense in \overline{Y} . \square

Corollary 2.2. *Let a Lie group G act smoothly on a smooth manifold M , M_{reg} be an open dense subset of M , and $M' \subset M_{\text{reg}}$ be a G -invariant constructible set. If there exists a smooth measure μ on M such that $\mu(M_{\text{reg}} \setminus M') = 0$, then M' contains a G -invariant open dense subset of M .*

Proof. Since μ is smooth and $\mu(M_{\text{reg}} \setminus M') = 0$, M' is dense in M_{reg} , and hence is dense in M . Since M' is constructible, by Lemma 2.1, M' contains an open dense subset U of M . But M' is G -invariant. So the G -invariant open dense subset $\bigcup_{g \in G} gU$ of M is contained in M' . \square

Lemma 2.3. *Let G and H be Lie groups with G connected, and let $G \times H$ act smoothly on a smooth manifold M . Consider the induced G -action on M/H . Then $(M/H)^G$ is constructible with respect to the quotient topology on M/H .*

Proof. It suffices to prove that the preimage P of $(M/H)^G$ under the quotient map $M \rightarrow M/H$ is of the form $\bigcup_{i=0}^k U_i \cap V_i$, where U_i is H -invariant and open, V_i is H -invariant and closed. If we denote $K = G \times H$, it is easy to see that

$$P = \{x \in M \mid Kx = Hx\}.$$

For $0 \leq i \leq \dim H$, we denote

$$U_i = \{x \in M \mid \dim Hx \geq i\}, \quad V_i = \{x \in M \mid \dim Kx \leq i\}.$$

Then U_i is H -invariant and open, V_i is H -invariant and closed. We prove the lemma by showing that $P = \bigcup_{i=0}^{\dim H} U_i \cap V_i$. It is obvious that $P \subset \bigcup_{i=0}^{\dim H} U_i \cap V_i$. To prove the converse, suppose that $x \in U_i \cap V_i$ for some i . Let $p_1 : K \rightarrow G$ be the projection. Then

$$\begin{aligned} \dim p_1(K(x)) &= \dim K(x) - \dim H(x) \\ &= (\dim K - \dim Kx) - (\dim H - \dim Hx) \\ &\geq (\dim K - i) - (\dim H - i) = \dim G. \end{aligned}$$

Since G is connected, we must have $p_1(K(x)) = G$. This means that $H \cdot K(x) = K$. Thus $Kx = Hx$, and hence $x \in P$. This completes the proof. \square

Remark 2.1. The set $(M/H)^G$ need not be locally closed. For example, if $G = H = \text{GL}(n, \mathbb{C})$, M is the space of $n \times n$ complex matrices, and $G \times H$ acts on M

by $(g, h)A = gAh^{-1}$, then the preimage of $(M/H)^G$ under $M \rightarrow M/H$ is equal to $\text{GL}(n, \mathbb{C}) \cup \{0\}$, which is not locally closed. Hence $(M/H)^G$ is not locally closed.

2.2. G -saturated sets. Let a topological group G act continuously on a topological space X . A subset $Y \subset X$ is G -saturated if $x \in X$, $y \in Y$ and $\overline{Gx} = \overline{Gy}$ imply that $x \in Y$. Obviously, G -saturated sets are G -invariant.

Lemma 2.4. *Let G and X be as above. Then the family of G -saturated subsets of X is closed under intersection, union, and complement.*

Proof. We define an equivalence relation on M by setting $x \sim y$ if and only if $\overline{Gx} = \overline{Gy}$. Then a subset of X is G -saturated if and only if it is the union of some equivalence classes. Hence the lemma follows. \square

Lemma 2.5. *Let G and X be as above. Then any G -invariant locally closed subset of X is G -saturated.*

Proof. Obviously, G -invariant closed sets are G -saturated. Then Lemma 2.4 implies that G -invariant open sets are also G -saturated. By Lemma 2.4 again, it suffices to prove that any G -invariant locally closed set $Y \subset X$ is the intersection of a G -invariant open set and a G -invariant closed set. Suppose that $Y = U \cap V$, where U is open, V is closed. Let $U' = \bigcup_{g \in G} gU$, $V' = \bigcap_{g \in G} gV$. Then U' is G -invariant and open, V' is G -invariant and closed. We claim that $Y = U' \cap V'$. Indeed, since Y is G -invariant, we have

$$Y = \bigcap_{g \in G} gY = \bigcap_{g \in G} (gU \cap gV) = \left(\bigcap_{g \in G} gU \right) \cap \left(\bigcap_{g \in G} gV \right) \subset U' \cap V'$$

and

$$Y = \bigcup_{g \in G} gY = \bigcup_{g \in G} (gU \cap gV) \supset \bigcup_{g \in G} (gU \cap V') = \left(\bigcup_{g \in G} gU \right) \cap V' = U' \cap V'.$$

This completes the proof. \square

Lemma 2.6. *Let a topological group G act continuously on two topological spaces X_1 and X_2 , and let $\theta : X_1 \rightarrow X_2$ be a G -equivariant continuous map. If $Y \subset X_2$ is G -saturated, then so is $\theta^{-1}(Y)$.*

Proof. Suppose $x \in X_1$, $y \in \theta^{-1}(Y)$ and $\overline{Gx} = \overline{Gy}$. Since Y is G -saturated, $\theta(y) \in Y$, and

$$\overline{G\theta(x)} = \overline{\theta(Gx)} = \overline{\theta(\overline{Gx})} = \overline{\theta(\overline{Gy})} = \overline{\theta(Gy)} = \overline{G\theta(y)},$$

we have $\theta(x) \in Y$. Hence $x \in \theta^{-1}(Y)$. This completes the proof. \square

Lemma 2.7. *Let a Lie group G act smoothly on a compact smooth manifold M , $M_0 \subset M$ be a G -minimal set, and μ be a G -invariant Borel measure on M supported on M_0 . If $M' \subset M$ is a G -saturated Borel set with $\mu(M') > 0$, then $M_0 \subset M'$.*

Proof. Since μ is supported on M_0 and $\mu(M') > 0$, we have $M_0 \cap M' \neq \emptyset$. Choose $x_0 \in M_0 \cap M'$, and let $x \in M_0$. Since M_0 is G -minimal, we have $\overline{Gx} = \overline{Gx_0}$. So $x \in M'$. Hence $M_0 \subset M'$. \square

2.3. Subquotients of Lie algebras. For simplicity, we assume that all Lie algebras in this subsection are real and finite-dimensional.

Lemma 2.8. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be Lie algebras. If $\mathfrak{a} \prec \mathfrak{b}$ and $\mathfrak{b} \prec \mathfrak{c}$, then $\mathfrak{a} \prec \mathfrak{c}$.*

Proof. By definition, there exist $\mathfrak{b}_1 < \mathfrak{b}$, $\mathfrak{c}_1 < \mathfrak{c}$, and surjective homomorphisms $\varphi : \mathfrak{b}_1 \rightarrow \mathfrak{a}$, $\psi : \mathfrak{c}_1 \rightarrow \mathfrak{b}$. Let $\mathfrak{c}_2 = \psi^{-1}(\mathfrak{b}_1)$. Then $\varphi \circ \psi|_{\mathfrak{c}_2} : \mathfrak{c}_2 \rightarrow \mathfrak{a}$ is a surjective homomorphism. Thus $\mathfrak{a} \prec \mathfrak{c}$. \square

Lemma 2.9. *Let $\mathfrak{a}, \mathfrak{b}$ be Lie algebras. If $\mathfrak{a} \prec \mathfrak{b}$, then $\text{ad}(\mathfrak{a}) \prec \text{ad}(\mathfrak{b})$.*

Proof. Let $\varphi : \mathfrak{b}_1 \rightarrow \mathfrak{a}$ be a surjective homomorphism, where $\mathfrak{b}_1 < \mathfrak{b}$. Since $\varphi(Z(\mathfrak{b}_1)) \subset Z(\mathfrak{a})$, we have

$$\begin{aligned} \text{ad}(\mathfrak{a}) &\cong \mathfrak{a}/Z(\mathfrak{a}) \prec \mathfrak{a}/\varphi(Z(\mathfrak{b}_1)) \cong \mathfrak{b}_1/\varphi^{-1}(\varphi(Z(\mathfrak{b}_1))) \\ &\prec \mathfrak{b}_1/Z(\mathfrak{b}_1) \prec \mathfrak{b}_1/(Z(\mathfrak{b}) \cap \mathfrak{b}_1) < \mathfrak{b}/Z(\mathfrak{b}) \cong \text{ad}(\mathfrak{b}). \end{aligned}$$

This completes the proof. \square

Lemma 2.10. *Let \mathfrak{f} be a Lie algebra, and let $\mathfrak{g}, \mathfrak{h} < \mathfrak{f}$ be such that $[\mathfrak{g}, \mathfrak{h}] = 0$ and $\mathfrak{f} = \mathfrak{g} + \mathfrak{h}$. Let V be a real vector space, $\ell : \mathfrak{f} \rightarrow V$ be a linear map such that $\ker(\ell) < \mathfrak{f}$ and $\mathfrak{f} = \mathfrak{h} + \ker(\ell)$. Then there exist $\mathfrak{z} < \mathfrak{h}$ and a Lie algebra structure on $\ell(\mathfrak{g})$ such that $\ell(\mathfrak{z}) = \ell(\mathfrak{g})$, and such that $\ell|_{\mathfrak{g}} : \mathfrak{g} \rightarrow \ell(\mathfrak{g})$ and $-\ell|_{\mathfrak{z}} : \mathfrak{z} \rightarrow \ell(\mathfrak{g})$ are Lie algebra homomorphisms. In particular, we have $\ell(\mathfrak{g}) \prec \mathfrak{h}$.*

Proof. Let $\mathfrak{k} = \ker(\ell)$, $\mathfrak{z} = \mathfrak{h} \cap (\mathfrak{g} + \mathfrak{k})$. Then $\ell(\mathfrak{z}) \subset \ell(\mathfrak{g} + \mathfrak{k}) = \ell(\mathfrak{g})$. On the other hand, it is easy to see that $\mathfrak{g} = \mathfrak{g} \cap (\mathfrak{h} + \mathfrak{k}) \subset \mathfrak{z} + \mathfrak{k}$. So $\ell(\mathfrak{g}) \subset \ell(\mathfrak{z} + \mathfrak{k}) = \ell(\mathfrak{z})$. Hence $\ell(\mathfrak{z}) = \ell(\mathfrak{g})$.

We claim that $\mathfrak{g} \cap \mathfrak{k} \triangleleft \mathfrak{f}$. Indeed, since $[\mathfrak{g}, \mathfrak{h}] = 0$ and $\mathfrak{f} = \mathfrak{g} + \mathfrak{h}$, we have $\mathfrak{g} \triangleleft \mathfrak{f}$, and hence $\mathfrak{g} \cap \mathfrak{k} \triangleleft \mathfrak{f}$. Thus $[\mathfrak{f}, \mathfrak{g} \cap \mathfrak{k}] = [\mathfrak{h}, \mathfrak{g} \cap \mathfrak{k}] + [\mathfrak{k}, \mathfrak{g} \cap \mathfrak{k}] \subset \mathfrak{g} \cap \mathfrak{k}$. This verifies the claim. Note that $\ker(\ell|_{\mathfrak{g}}) = \mathfrak{g} \cap \mathfrak{k}$. So there exists a Lie algebra structure on $\ell(\mathfrak{g})$ such that $\ell|_{\mathfrak{g}}$ is a homomorphism. It remains to prove that $-\ell|_{\mathfrak{z}}$ is a homomorphism. Let $Z_i \in \mathfrak{z}$, $i = 1, 2$. Then $Z_i = X_i + Y_i$ for some $X_i \in \mathfrak{g}$ and $Y_i \in \mathfrak{k}$. Since

$$\begin{aligned} [Z_1, Z_2] &= [X_1, X_2] + [Y_1, Y_2] + [X_1, Y_2] + [Y_1, X_2] \\ &= [X_1, X_2] + [Y_1, Y_2] + [X_1, Z_2 - X_2] + [Z_1 - X_1, X_2] \\ &= -[X_1, X_2] + [Y_1, Y_2], \end{aligned}$$

we have

$$-\ell([Z_1, Z_2]) = \ell([X_1, X_2]) = [\ell(X_1), \ell(X_2)] = [-\ell(Z_1), -\ell(Z_2)].$$

Thus $-\ell|_{\mathfrak{z}}$ is a homomorphism. \square

3. REVIEW OF GEOMETRIC STRUCTURES

In this section, we briefly review some definitions and facts in Gromov's theory of rigid geometric structures [21]. Let n and r be positive integers. We denote by $\text{GL}^r(n)$ the real algebraic group of r -jets at $0 \in \mathbb{R}^n$ of diffeomorphisms of \mathbb{R}^n that fix 0. Let M be a connected n -dimensional C^ε manifold, where $\varepsilon = \infty$ or ω . We denote by $F^r(M)$ the r -th order frame bundle of M . This is a principal $\text{GL}^r(n)$ -bundle over M , whose fiber $F^r(M)_x$ at $x \in M$ consists of the r -jets at $0 \in \mathbb{R}^n$ of diffeomorphisms from some neighborhood of 0 in \mathbb{R}^n into M that send 0 to x . Let V be a topological space, and let $\text{GL}^r(n)$ act continuously on V . A continuous geometric structure of order r and type V on M is a continuous

$\mathrm{GL}^r(n)$ -equivariant map $\sigma : F^r(M) \rightarrow V$. We refer to the induced continuous map $\theta : M \rightarrow V/\mathrm{GL}^r(n)$ as the *Gauss map* of σ , where $V/\mathrm{GL}^r(n)$ is endowed with the quotient topology. We say that σ is C^ε if V is a C^ε $\mathrm{GL}^r(n)$ -manifold and σ is a C^ε map, and say that σ is a *structure of algebraic type* (A-structure, for short) if V is a real algebraic manifold and the $\mathrm{GL}^r(n)$ -action on V is regular. Whenever σ is C^ε , we say that it is *unimodular* if there is a preassigned $\mathrm{GL}^r(n)$ -equivariant C^ε map $\delta : V \rightarrow (0, +\infty)$, where $\mathrm{GL}^r(n)$ acts on $(0, +\infty)$ by $\alpha.t = |\det(\pi_1^r(\alpha))|t$, here $\alpha \in \mathrm{GL}^r(n)$, $t \in (0, +\infty)$, and $\pi_1^r : \mathrm{GL}^r(n) \rightarrow \mathrm{GL}(n, \mathbb{R})$ is the natural projection. The composition $\delta \circ \sigma$ induces a C^ε volume density $F^1(M) \rightarrow (0, +\infty)$, and hence a smooth measure on M .

Now we assume that $\sigma : F^r(M) \rightarrow V$ is a C^ε geometric structure. Let $s \geq 0$ be an integer, and let $J_n^s(V)$ denote the space of s -jets at $0 \in \mathbb{R}^n$ of C^ε maps from some neighborhood of 0 in \mathbb{R}^n into V . Then $\mathrm{GL}^{r+s}(n)$ acts naturally on $J_n^s(V)$. The s -th *prolongation* $\sigma^s : F^{r+s}(M) \rightarrow J_n^s(V)$ of σ is a C^ε geometric structure of order $r+s$ (see [9, 16]). If σ is an A-structure, then so is σ^s .

If f is a C^ε map from a neighborhood of $x \in M$ to some manifold or is the germ at x of such a map, we denote by $j_x^i f$ the i -jet of f at x , where $i \geq 0$ is an integer. Let f be a C^ε diffeomorphism from some open set $U \subset M$ into M . We say that f is a *local C^ε isometry* if $\sigma(j_x^r f \circ \beta) = \sigma(\beta)$ for all $x \in U$ and $\beta \in F^r(M)_x$. If moreover $U = f(U) = M$, then f is called a *C^ε isometry*. Note that if σ is unimodular, then the induced smooth measure on M is preserved by the C^ε isometry group $\mathrm{Iso}(M)$. For $i \geq r$ and $x \in U$, if $\sigma^{i-r}(j_x^i f \circ \beta) = \sigma^{i-r}(\beta)$ for all $\beta \in F^i(M)_x$, then $j_x^i f$ is called an *infinitesimal isometry* of order i . We denote by $\mathrm{Iso}^{\mathrm{germ}}(M)$ and $\mathrm{Iso}^i(M)$ the groupoids of germs of local C^ε isometries and infinitesimal isometries of order i respectively, denote by $\mathrm{Iso}_{x,y}^{\mathrm{germ}}(M)$ (resp. $\mathrm{Iso}_{x,y}^i(M)$) the subset of $\mathrm{Iso}^{\mathrm{germ}}(M)$ (resp. $\mathrm{Iso}^i(M)$) consisting of elements with source x and target y , and denote $\mathrm{Iso}_x^{\mathrm{germ}}(M) = \mathrm{Iso}_{x,x}^{\mathrm{germ}}(M)$, $\mathrm{Iso}_x^i(M) = \mathrm{Iso}_{x,x}^i(M)$. Then we have natural maps $\mathrm{Iso}_{x,y}^{\mathrm{germ}}(M) \rightarrow \mathrm{Iso}_{x,y}^{i+1}(M) \rightarrow \mathrm{Iso}_{x,y}^i(M)$. We say that $x \in M$ is *k -regular* if there exists an open neighborhood U_x of x such that $\mathrm{Iso}_{x,y}^{\mathrm{germ}}(M) \rightarrow \mathrm{Iso}_{x,y}^k(M)$ is surjective for every $y \in U_x$.

If v is a C^ε local vector field on M defined around x , we denote the i -jet of v at x by $j_x^i v$. Let $\mathrm{Vect}_x^i(M)$ denote the space of all such i -jets at x . A (local) C^ε vector field v on M is called a (local) *C^ε Killing field* if the local flow generated by v consists of local C^ε isometries. Note that there are natural maps $\mathrm{Kill}(M) \rightarrow \mathrm{Kill}_x^{\mathrm{germ}}(M) \rightarrow \mathrm{Vect}_x^i(M)$, where $\mathrm{Kill}(M)$ and $\mathrm{Kill}_x^{\mathrm{germ}}(M)$ are the Lie algebras of C^ε Killing fields and germs at x of local C^ε Killing fields defined around x , respectively. We denote the kernel of $\mathrm{ev}_x : \mathrm{Kill}_x^{\mathrm{germ}}(M) \rightarrow T_x M$ by $\mathrm{Kill}_x^{\mathrm{germ}}(M)_0$, which is a subalgebra of $\mathrm{Kill}_x^{\mathrm{germ}}(M)$.

The C^ε geometric structure σ is *i -rigid* if $\mathrm{Iso}_x^{i+1}(M) \rightarrow \mathrm{Iso}_x^i(M)$ is injective for every $x \in M$, and is *rigid* if it is i -rigid for some $i \geq r$. Rigid geometric structures have the following properties.

Theorem 3.1 (Gromov). *Let M be a connected C^ε manifold, and let σ be an i -rigid C^ε geometric structure of order r on M , where $i \geq r > 0$. Then we have*

- (1) σ is i' -rigid for any $i' \geq i$.
- (2) For any $x \in M$, $\mathrm{Iso}_x^{\mathrm{germ}}(M) \rightarrow \mathrm{Iso}_x^i(M)$ is injective.
- (3) For any $x \in M$, $\mathrm{Kill}(M) \rightarrow \mathrm{Kill}_x^{\mathrm{germ}}(M)$ and $\mathrm{Kill}_x^{\mathrm{germ}}(M) \rightarrow \mathrm{Vect}_x^i(M)$ are injective. In particular, $\mathrm{Kill}(M)$ and $\mathrm{Kill}_x^{\mathrm{germ}}(M)$ are finite-dimensional.

- (4) $\text{Iso}(M)$ is a Lie group and its action on M is C^ε . Moreover, for any $x \in M$, $\text{Iso}_x^{\text{germ}}(M)$ is a Lie group.
- (5) There exist an $\text{Iso}^{\text{germ}}(M)$ -invariant open dense subset M_{reg} of M and an integer $k > i$ such that every point in M_{reg} is k -regular.

Proof. Detailed proof of (1) can be found in [9, 16]. (2) is implied from [21, 1.2.A and 1.5.A]. We sketch a proof of (3) using (2). If v is a local Killing field defined around x with $j_x^i v = 0$, and if f_t is the local flow generated by v , then by [9, Lem. 2.1], we have $j_x^i f_t = j_x^i \text{id}_M$ for every t . Thus (2) implies that $f_t = \text{id}_M$ on some neighborhood U_t of x . Hence $\langle v \rangle_x = 0$. This proves the injectivity of $\text{Kill}_x^{\text{germ}}(M) \rightarrow \text{Vect}_x^i(M)$. Now if $w \in \text{Kill}(M)$ with $\langle w \rangle_x = 0$, then the set $Z = \{y \in M \mid j_y^i w = 0\}$ is closed and nonempty. By the injectivity of $\text{Kill}_y^{\text{germ}}(M) \rightarrow \text{Vect}_y^i(M)$, Z is also open. So $Z = M$, and hence $w = 0$. This proves the injectivity of $\text{Kill}(M) \rightarrow \text{Kill}_x^{\text{germ}}(M)$. (4) is proved in [21, Sect. 1.6.H]. (5) is proved in [21, Thm. 1.6.F] and [3, 9, 16, 36, 37]. \square

Note that $\mathcal{L}(\text{Iso}(M))$ can be identified with the Lie algebra of complete Killing fields on M via the induced infinitesimal action. Under this identification, if $v \in \text{Kill}(M)$ is complete, then $t \mapsto \exp(-tv)$ is the flow generated by v . Similarly, for $x \in M$, $\mathcal{L}(\text{Iso}_x^{\text{germ}}(M))$ can be identified with $\text{Kill}_x^{\text{germ}}(M)_0$ in such a way that for a local Killing field v which is defined around x and vanishes at x , if f_t is the local flow generated by v , then $\exp(-t\langle v \rangle_x) = \langle f_t \rangle_x$.

The next theorem holds only for analytic structures.

Theorem 3.2 (Gromov). *Let M be a connected analytic manifold with an i -rigid analytic A-structure.*

- (1) *If M is compact, then there exists $k > i$ such that every point in M is k -regular.*
- (2) *If M is simply connected, then for any $x \in M$, $\text{Kill}(M) \rightarrow \text{Kill}_x^{\text{germ}}(M)$ is bijective.*

Proof. (1) is proved in [21, Sect. 1.7.B], and (2) is proved in [21, Sect. 1.7]. \square

Finally, we remark that a geometric structure $\sigma : F^r(M) \rightarrow V$ naturally induces a geometric structure $\tilde{\sigma} : F^r(\tilde{M}) \rightarrow V$ on \tilde{M} by $\tilde{\sigma}(\tilde{\beta}) = \sigma(j_{\tilde{x}}^r \pi \circ \tilde{\beta})$, where $\tilde{x} \in \tilde{M}$ and $\tilde{\beta} \in F^r(\tilde{M})_{\tilde{x}}$. If σ is a rigid C^ε A-structure, then so is $\tilde{\sigma}$. If a connected Lie group G acts on M by C^ε isometries, then the induced C^ε action of \tilde{G} on \tilde{M} is also isometric.

4. ZARISKI HULLS OF FUNDAMENTAL GROUPS

Let M be a connected C^ε manifold, and let $\Gamma = \pi_1(M)$. We denote by $\text{Diff}^{\text{germ}}(M)$ the groupoid of germs of local C^ε diffeomorphisms of M , by $\text{Diff}_{x,y}^{\text{germ}}(M)$ the subset of $\text{Diff}^{\text{germ}}(M)$ consisting of germs with source x and target y , by $\text{Vect}(M)$ the Lie algebra of C^ε vector fields on M , and by $\text{Vect}_x^{\text{germ}}(M)$ the Lie algebra of germs at x of local C^ε vector fields defined around x . For $\tilde{x} \in \tilde{M}$, we denote by

$$(4.1) \quad \lambda_{\tilde{x}} : \text{Vect}(\tilde{M}) \rightarrow \text{Vect}_x^{\text{germ}}(M)$$

the composition of the natural homomorphism $\text{Vect}(\tilde{M}) \rightarrow \text{Vect}_{\tilde{x}}^{\text{germ}}(\tilde{M})$ and the natural isomorphism $\text{Vect}_{\tilde{x}}^{\text{germ}}(\tilde{M}) \rightarrow \text{Vect}_x^{\text{germ}}(M)$, where $x = \pi(\tilde{x})$. In other words, for $w \in \text{Vect}(\tilde{M})$, we have $\lambda_{\tilde{x}}(w) = d\pi(\langle w \rangle_{\tilde{x}})$. The deck transformations

induce a representation of Γ in $\text{Vect}(\widetilde{M})$ by taking differentials. If $\mathcal{W} \subset \text{Vect}(\widetilde{M})$ is a finite-dimensional Γ -invariant subspace, we denote the restricted representation in \mathcal{W} by $\rho_{\mathcal{W}} : \Gamma \rightarrow \text{GL}(\mathcal{W})$.

Definition 4.1. Let \mathcal{W} be a finite-dimensional Γ -invariant subspace of $\text{Vect}(\widetilde{M})$.

- (1) Let $x, y \in M$, $\mathcal{A}_{x,y} \subset \text{Diff}_{x,y}^{\text{germ}}(M)$. Choose $\tilde{x} \in \pi^{-1}(x)$, $\tilde{y} \in \pi^{-1}(y)$. The *Zariski hull of Γ in $\mathcal{A}_{x,y}$ relative to \mathcal{W}* is defined as

$$\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_{x,y}) = \{\varphi \in \mathcal{A}_{x,y} \mid \text{there exists } A \in \overline{\rho_{\mathcal{W}}(\Gamma)} \text{ such that} \\ d\varphi \circ \lambda_{\tilde{x}}|_{\mathcal{W}} = \lambda_{\tilde{y}} \circ A\}.$$

- (2) Let \mathcal{A} be a subgroupoid of $\text{Diff}^{\text{germ}}(M)$. The *Zariski hull of Γ in \mathcal{A} relative to \mathcal{W}* is defined as

$$\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}) = \bigcup_{x,y \in M} \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A} \cap \text{Diff}_{x,y}^{\text{germ}}(M)).$$

- (3) Let $x \in M$, $\mathfrak{a}_x < \text{Vect}_x^{\text{germ}}(M)$. Choose $\tilde{x} \in \pi^{-1}(x)$. The *Zariski hull of Γ in \mathfrak{a}_x relative to \mathcal{W}* is defined as

$$\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x) = \{\eta \in \mathfrak{a}_x \mid \text{there exists } B \in \mathcal{L}(\overline{\rho_{\mathcal{W}}(\Gamma)}) \text{ such that} \\ \text{ad}(\eta) \circ \lambda_{\tilde{x}}|_{\mathcal{W}} = \lambda_{\tilde{x}} \circ B\},$$

where we view $\text{ad}(\eta) : \text{Vect}_x^{\text{germ}}(M) \rightarrow \text{Vect}_x^{\text{germ}}(M)$.

It is obvious that if $\mathcal{W} = 0$, then $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}) = \mathcal{A}$ and $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x) = \mathfrak{a}_x$. But to justify Definition 4.1 for general \mathcal{W} , we need the following lemma.

Lemma 4.1. For $\tilde{x} \in \widetilde{M}$ and $\gamma \in \Gamma$, we have $\lambda_{\gamma\tilde{x}} = \lambda_{\tilde{x}} \circ d\gamma^{-1}$.

Proof. Since $\pi = \pi \circ \gamma^{-1}$, we have

$$\lambda_{\gamma\tilde{x}}(w) = d\pi(\langle w \rangle_{\gamma\tilde{x}}) = d\pi(d\gamma^{-1}(\langle w \rangle_{\tilde{x}})) = d\pi(\langle d\gamma^{-1}(w) \rangle_{\tilde{x}}) = \lambda_{\tilde{x}}(d\gamma^{-1}(w))$$

for all $w \in \text{Vect}(\widetilde{M})$. \square

Lemma 4.2. Under the situation of Definition 4.1, we have

- (1) $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_{x,y})$ is independent of the choices of \tilde{x} and \tilde{y} , and hence $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A})$ is well-defined.
(2) $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x)$ is independent of the choice of \tilde{x} .

Proof. (1) If $\tilde{x}' \in \pi^{-1}(x)$ and $\tilde{y}' \in \pi^{-1}(y)$, then there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $\tilde{x}' = \gamma_1\tilde{x}$ and $\tilde{y}' = \gamma_2\tilde{y}$. Let $\varphi \in \mathcal{A}_{x,y}$. Suppose that there exists $A \in \overline{\rho_{\mathcal{W}}(\Gamma)}$ such that $d\varphi \circ \lambda_{\tilde{x}}|_{\mathcal{W}} = \lambda_{\tilde{y}} \circ A$. Then by Lemma 4.1, we have

$$d\varphi \circ \lambda_{\tilde{x}'}|_{\mathcal{W}} = d\varphi \circ \lambda_{\tilde{x}} \circ \rho_{\mathcal{W}}(\gamma_1)^{-1} = \lambda_{\tilde{y}} \circ A \rho_{\mathcal{W}}(\gamma_1)^{-1} \\ = \lambda_{\tilde{y}'} \circ \rho_{\mathcal{W}}(\gamma_2) A \rho_{\mathcal{W}}(\gamma_1)^{-1}.$$

Since $\rho_{\mathcal{W}}(\gamma_2) A \rho_{\mathcal{W}}(\gamma_1)^{-1} \in \overline{\rho_{\mathcal{W}}(\Gamma)}$, this proves (1).

(2) Let $\gamma \in \Gamma$ and $\eta \in \mathfrak{a}_x$. If there exists $B \in \mathcal{L}(\overline{\rho_{\mathcal{W}}(\Gamma)})$ such that $\text{ad}(\eta) \circ \lambda_{\tilde{x}}|_{\mathcal{W}} = \lambda_{\tilde{x}} \circ B$, then by Lemma 4.1, we have

$$\text{ad}(\eta) \circ \lambda_{\gamma\tilde{x}}|_{\mathcal{W}} = \text{ad}(\eta) \circ \lambda_{\tilde{x}} \circ \rho_{\mathcal{W}}(\gamma)^{-1} = \lambda_{\tilde{x}} \circ B \rho_{\mathcal{W}}(\gamma)^{-1} \\ = \lambda_{\gamma\tilde{x}} \circ \rho_{\mathcal{W}}(\gamma) B \rho_{\mathcal{W}}(\gamma)^{-1}.$$

Since $\rho_{\mathcal{W}}(\gamma)B\rho_{\mathcal{W}}(\gamma)^{-1} \in \mathcal{L}\left(\overline{\rho_{\mathcal{W}}(\Gamma)}\right)$, this proves (2). \square

Lemma 4.3. (1) *Under the situation of Definition 4.1(2), $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A})$ is a subgroupoid of \mathcal{A} .*

(2) *Under the situation of Definition 4.1(3), $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x)$ is a subalgebra of \mathfrak{a}_x .*

Proof. (1) It suffices to prove that if $\varphi_1 \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_{x,y})$ and $\varphi_2 \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_{x,z})$, then $\varphi_2 \circ \varphi_1^{-1} \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_{y,z})$. Let $\tilde{x} \in \pi^{-1}(x)$, $\tilde{y} \in \pi^{-1}(y)$, $\tilde{z} \in \pi^{-1}(z)$, and let $A_1, A_2 \in \overline{\rho_{\mathcal{W}}(\Gamma)}$ be such that $d\varphi_1 \circ \lambda_{\tilde{x}}|_{\mathcal{W}} = \lambda_{\tilde{y}} \circ A_1$ and $d\varphi_2 \circ \lambda_{\tilde{x}}|_{\mathcal{W}} = \lambda_{\tilde{z}} \circ A_2$. Then

$$d(\varphi_2 \circ \varphi_1^{-1}) \circ \lambda_{\tilde{y}}|_{\mathcal{W}} = d\varphi_2 \circ \lambda_{\tilde{x}} \circ A_1^{-1} = \lambda_{\tilde{z}} \circ A_2 A_1^{-1}.$$

Since $A_2 A_1^{-1} \in \overline{\rho_{\mathcal{W}}(\Gamma)}$, we have $\varphi_2 \circ \varphi_1^{-1} \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_{y,z})$.

(2) Let $\eta_i \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x)$ ($i = 1, 2$), $\tilde{x} \in \pi^{-1}(x)$, and let $B_i \in \mathcal{L}\left(\overline{\rho_{\mathcal{W}}(\Gamma)}\right)$ be such that $\text{ad}(\eta_i) \circ \lambda_{\tilde{x}}|_{\mathcal{W}} = \lambda_{\tilde{x}} \circ B_i$. Then

$$\begin{aligned} \text{ad}([\eta_1, \eta_2]) \circ \lambda_{\tilde{x}}|_{\mathcal{W}} &= \text{ad}(\eta_1) \circ \text{ad}(\eta_2) \circ \lambda_{\tilde{x}}|_{\mathcal{W}} - \text{ad}(\eta_2) \circ \text{ad}(\eta_1) \circ \lambda_{\tilde{x}}|_{\mathcal{W}} \\ &= \text{ad}(\eta_1) \circ \lambda_{\tilde{x}} \circ B_2 - \text{ad}(\eta_2) \circ \lambda_{\tilde{x}} \circ B_1 \\ &= \lambda_{\tilde{x}} \circ B_1 B_2 - \lambda_{\tilde{x}} \circ B_2 B_1 \\ &= \lambda_{\tilde{x}} \circ [B_1, B_2]. \end{aligned}$$

Thus $[\eta_1, \eta_2] \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x)$. \square

Now we assume that M is endowed with a rigid C^{ε} geometric structure and $\mathcal{W} \subset \text{Kill}(\widetilde{M})$. For $x \in M$ and $\mathfrak{a}_x < \text{Kill}_x^{\text{germ}}(M)$, we denote

$$(4.2) \quad \mathfrak{a}_x^{\mathcal{W}} = \{\eta \in \mathfrak{a}_x \mid [\eta, \lambda_{\tilde{x}}(\mathcal{W})] = 0\},$$

where $\tilde{x} \in \pi^{-1}(x)$. Note that by Lemma 4.1, $\lambda_{\tilde{x}}(\mathcal{W})$ is independent of the choice of \tilde{x} . The following lemma gives a direct relation between $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x)$ and $\overline{\rho_{\mathcal{W}}(\Gamma)}$.

Lemma 4.4. *Let M be a connected C^{ε} manifold with a rigid C^{ε} geometric structure, \mathcal{W} be a Γ -invariant subspace of $\text{Kill}(\widetilde{M})$, $x \in M$, and $\mathfrak{a}_x < \text{Kill}_x^{\text{germ}}(M)$. Then there exists an exact sequence*

$$0 \rightarrow \mathfrak{a}_x^{\mathcal{W}} \hookrightarrow \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x) \rightarrow \mathcal{L}\left(\overline{\rho_{\mathcal{W}}(\Gamma)}\right)$$

of Lie algebra homomorphisms.

Proof. Let $\tilde{x} \in \pi^{-1}(x)$, and denote $\mathcal{W}_x = \lambda_{\tilde{x}}(\mathcal{W})$. By Theorem 3.1(3), the restriction of $\lambda_{\tilde{x}}$ to \mathcal{W} is injective. Hence it induces an isomorphism $\lambda'_{\tilde{x}} : \mathcal{W} \rightarrow \mathcal{W}_x$. Let $\mathfrak{a}'_x = \{\eta \in \mathfrak{a}_x \mid \text{ad}(\eta)(\mathcal{W}_x) \subset \mathcal{W}_x\}$. For $\eta \in \mathfrak{a}'_x$, we denote $\delta(\eta) = \lambda'^{-1}_{\tilde{x}} \circ \text{ad}(\eta) \circ \lambda'_{\tilde{x}}$. Then $\delta : \mathfrak{a}'_x \rightarrow \mathfrak{gl}(\mathcal{W})$ is a representation and $\ker(\delta) = \mathfrak{a}_x^{\mathcal{W}}$. It is easy to see that $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x) = \delta^{-1}\left(\mathcal{L}\left(\overline{\rho_{\mathcal{W}}(\Gamma)}\right)\right)$. Thus δ restricts to a homomorphism $\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x) \rightarrow \mathcal{L}\left(\overline{\rho_{\mathcal{W}}(\Gamma)}\right)$ with kernel $\mathfrak{a}_x^{\mathcal{W}}$. This completes the proof. \square

Recall that there is a natural identification $\mathcal{L}(\text{Iso}_x^{\text{germ}}(M)) = \text{Kill}_x^{\text{germ}}(M)_0$ such that if v is a local Killing field defined around x with $\text{ev}_x(v) = 0$ and f_t is the local flow generated by v , then $\exp(-t\langle v \rangle_x) = \langle f_t \rangle_x$. The next result will be used in the proof of Theorem 1.2(2).

Lemma 4.5. *Let M , \mathcal{W} , and x be as in Lemma 4.4, and let \mathcal{A}_x be a closed subgroup of $\text{Iso}_x^{\text{germ}}(M)$. Then $\text{Hull}_\Gamma^\mathcal{W}(\mathcal{A}_x)$ is a closed subgroup of \mathcal{A}_x and $\mathcal{L}(\text{Hull}_\Gamma^\mathcal{W}(\mathcal{A}_x)) = \text{Hull}_\Gamma^\mathcal{W}(\mathcal{L}(\mathcal{A}_x))$.*

Proof. Let \tilde{x} , \mathcal{W}_x and λ'_x be as in the proof of Lemma 4.4, and let $\mathcal{A}'_x = \{\varphi \in \mathcal{A}_x \mid d\varphi(\mathcal{W}_x) = \mathcal{W}_x\}$. Then \mathcal{A}'_x is a closed subgroup of \mathcal{A}_x . For $\varphi \in \mathcal{A}'_x$, we denote $\rho(\varphi) = \lambda'^{-1}_x \circ d\varphi \circ \lambda'_x$. Then $\rho : \mathcal{A}'_x \rightarrow \text{GL}(\mathcal{W})$ is a representation and $\text{Hull}_\Gamma^\mathcal{W}(\mathcal{A}_x) = \rho^{-1}\left(\overline{\rho_\mathcal{W}(\Gamma)}\right)$. Thus $\text{Hull}_\Gamma^\mathcal{W}(\mathcal{A}_x)$ is a closed subgroup of \mathcal{A}_x . Denote $\mathfrak{a}_x = \mathcal{L}(\mathcal{A}_x)$, and let \mathfrak{a}'_x and δ be as in the proof of Lemma 4.4. By taking Lie derivatives, we see that $\mathcal{L}(\mathcal{A}'_x) = \mathfrak{a}'_x$ and $d\rho = \delta$. Thus $\mathcal{L}(\text{Hull}_\Gamma^\mathcal{W}(\mathcal{A}_x)) = \delta^{-1}\left(\mathcal{L}\left(\overline{\rho_\mathcal{W}(\Gamma)}\right)\right) = \text{Hull}_\Gamma^\mathcal{W}(\mathfrak{a}_x)$. \square

5. DISCOMPACT GROUPS

Let \mathbb{k} be a locally compact non-discrete field. The notion of \mathbb{k} -discompact groups was introduced by Shalom [32] to distinguish those \mathbb{k} -groups for which the Borel density theorem holds. A \mathbb{k} -group \mathbf{G} is \mathbb{k} -compact if $\mathbf{G}_\mathbb{k}$ is compact, and is \mathbb{k} -discompact if any \mathbb{k} -homomorphism from \mathbf{G} to a \mathbb{k} -compact \mathbb{k} -group is trivial. Obviously, \mathbb{k} -discompact groups are connected. It was proved in [32] that any \mathbb{k} -group \mathbf{G} admits a unique maximal \mathbb{k} -discompact \mathbb{k} -subgroup $R_d(\mathbf{G})$ of \mathbf{G} , called the \mathbb{k} -discompact radical of \mathbf{G} , which is characteristic for all \mathbb{k} -automorphisms of \mathbf{G} , such that $\mathbf{G}/R_d(\mathbf{G})$ is \mathbb{k} -compact.

If $\text{char } \mathbb{k} = 0$, the situation becomes much simpler. Recall that a normal \mathbb{k} -subgroup \mathbf{N} of a \mathbb{k} -group \mathbf{G} is \mathbb{k} -cocompact if $\mathbf{G}_\mathbb{k}/\mathbf{N}_\mathbb{k}$ is compact. If $\text{char } \mathbb{k} = 0$, then \mathbf{N} is \mathbb{k} -cocompact if and only if \mathbf{G}/\mathbf{N} is \mathbb{k} -compact ([32, Sect. 2.5]). Hence \mathbf{G} is \mathbb{k} -discompact if and only if it does not admit proper normal \mathbb{k} -cocompact \mathbb{k} -subgroups. This enables us to characterize the \mathbb{k} -discompactness of \mathbf{G} as follows.

Proposition 5.1. *Let \mathbf{G} be a connected \mathbb{k} -group, where $\text{char } \mathbb{k} = 0$. Then \mathbf{G} is \mathbb{k} -discompact if and only if the radical $R(\mathbf{G})$ is \mathbb{k} -split and all almost \mathbb{k} -simple factors of the semisimple \mathbb{k} -group $\mathbf{G}/R(\mathbf{G})$ are \mathbb{k} -isotropic.*

Proof. It follows from [32, Thm. 3.6 and Prop. 3.12] and [29, Thm. 3.1]. \square

Let $G = \mathbf{G}_\mathbb{R}$ be a real algebraic group, where \mathbf{G} is an \mathbb{R} -group. By taking the Zariski closure of G in \mathbf{G} , we may assume without loss of generality that G is Zariski dense in \mathbf{G} . We say that G is *discompact* if \mathbf{G} is \mathbb{R} -discompact. Then G is *discompact* if and only if it has no proper normal cocompact algebraic subgroups. Note that if G is *discompact* then it is Zariski connected. As in [2], we refer to $R_d(G) = R_d(\mathbf{G})_\mathbb{R}$ as the *discompact radical* of G , which is the unique maximal *discompact* algebraic subgroups of G and is characteristic (for algebraic automorphisms of G) and cocompact in G . It is obvious that $R(G) = (R(\mathbf{G})_\mathbb{R})_0$, where $R(G)$ is the Lie-theoretic radical of G . We say that $R(G)$ is \mathbb{R} -split if so is $R(\mathbf{G})$.

Corollary 5.2. *Let G be a Zariski connected real algebraic group. Then G is *discompact* if and only if $R(G)$ is \mathbb{R} -split and $G_0/R(G)$ has no compact factors.*

Proof. It follows from Proposition 5.1 and the fact that $G_0/R(G)$ is locally isomorphic to $(\mathbf{G}/R(\mathbf{G}))_\mathbb{R}$. \square

In the rest of this section, we study the discreteness of a Lie group G with respect to a (finite-dimensional) real representation $\rho : G \rightarrow \mathrm{GL}(\mathcal{V})$. We will make extensive use of Levi subgroups of G , which are maximal connected semisimple Lie subgroups of G and satisfy the following properties.

- All Levi subgroups of G are mutually conjugate and are locally isomorphic to $G_0/R(G)$. If L is one of them, then $G_0 = L \cdot R(G)$ ([35]).
- If L is a connected semisimple Lie subgroup of G , N is a connected solvable normal Lie subgroup of G , and $G_0 = L \cdot N$, then L is a Levi subgroup of G and $N = R(G)$ ([1, Lem. 4.9.1]).
- If $\varphi : G \rightarrow H$ is a homomorphism of Lie groups and L is a Levi subgroup of G , then $\varphi(L)$ is a Levi subgroup of $\varphi(G)$ and $R(\varphi(G)) = \varphi(R(G))$ ([1, Lem. 5.5.1]).

Proposition 5.3. *Let G be a connected Lie group, and let $\rho : G \rightarrow \mathrm{GL}(\mathcal{V})$ be a representation of G in a real vector space \mathcal{V} . Then G is ρ -discompact if and only if*

- (1) *all elements in $\rho(R(G))$ have only real eigenvalues, and*
- (2) *any Levi subgroup of $\rho(G)$ has no compact factors.*

Proof. Since $\rho(R(G)) = R(\rho(G))$, we may assume without loss of generality that G is a connected Lie subgroup of $\mathrm{GL}(\mathcal{V})$ and ρ is the inclusion map. Let L be a Levi subgroup of G . Then $G = L \cdot R(G)$. Since $(\overline{L})_0 = L$, we have $(\overline{G})_0 = L \cdot (\overline{R(G)})_0$ (see [27, Prop. 4.6.4]). This implies that (i) L is a Levi subgroup of \overline{G} , and (ii) $R(\overline{G}) = (\overline{R(G)})_0$. Assertion (i) implies that L is locally isomorphic to $(\overline{G})_0 / R(\overline{G})$. Hence L has no compact factors if and only if so does $(\overline{G})_0 / R(\overline{G})$. We also have

$$\begin{aligned}
 & \text{all eigenvalues of every } g \in R(G) \text{ are real} \\
 \iff & R(G) \text{ fixes a full flag of } \mathcal{V} && \text{(by [22, Cor. 1.29])} \\
 \iff & (\overline{R(G)})_0 \text{ fixes a full flag of } \mathcal{V} \\
 \iff & R(\overline{G}) \text{ fixes a full flag of } \mathcal{V} && \text{(by assertion (ii))} \\
 \iff & R(\overline{G}) \text{ is } \mathbb{R}\text{-split} && \text{(by [8, Thm. 15.4(iii)])}.
 \end{aligned}$$

Now the proposition follows from Corollary 5.2. \square

The discreteness of G with respect to its adjoint representation is of particular interest to us (this corresponds to the case of Lemma 1.1 where G acts faithfully on M and $\mathcal{V} = \mathfrak{g}$). Proposition 5.3 implies the following characterization of the Ad-discreteness of G .

Corollary 5.4. *Let G be a connected Lie group. Then G is Ad-discompact if and only if $R(G)$ is split solvable and $G/R(G)$ has no compact factors.*

Proof. By Proposition 5.3, G is Ad-discompact if and only if (i) all eigenvalues of $\mathrm{Ad}(g)$ are real for every $g \in R(G)$, and (ii) any Levi subgroup of $\mathrm{Ad}(G)$ has no compact factors. Since $R(G)$ is normal in G , any subspace of \mathfrak{g} containing $\mathcal{L}(R(G))$ is $R(G)$ -invariant. So for $g \in R(G)$, all eigenvalues of $\mathrm{Ad}(g)$ are real if and only if

all eigenvalues of $\text{Ad}(g)|_{\mathcal{L}(R(G))}$ are real. Hence assertion (i) holds if and only if $R(G)$ is split solvable. On the other hand, any Levi subgroup of $\text{Ad}(G)$ is locally isomorphic to $G/R(G)$. So assertion (ii) holds if and only if $G/R(G)$ has no compact factors. Hence the corollary follows. \square

Now we use Corollary 5.4 to prove Lemma 1.1. We need the following lemma.

Lemma 5.5. *Let $\rho : G \rightarrow \text{GL}(\mathcal{V})$ be a representation of a Lie group G in a real vector space \mathcal{V} , let $\mathcal{V}_0 \subset \mathcal{V}$ be a G -invariant subspace, and let $\rho_1 : G \rightarrow \text{GL}(\mathcal{V}_0)$, $\rho_2 : G \rightarrow \text{GL}(\mathcal{V}/\mathcal{V}_0)$ be the subrepresentation and quotient representation, respectively.*

- (1) *If G is ρ -discompact, then it is ρ_1 -discompact and ρ_2 -discompact.*
- (2) *If G is connected, then the converse of (1) also holds.*

Proof. Denote $\text{GL}(\mathcal{V}, \mathcal{V}_0) = \{A \in \text{GL}(\mathcal{V}) \mid A(\mathcal{V}_0) = \mathcal{V}_0\}$, and consider the algebraic homomorphisms $\varphi_1 : \text{GL}(\mathcal{V}, \mathcal{V}_0) \rightarrow \text{GL}(\mathcal{V}_0)$ and $\varphi_2 : \text{GL}(\mathcal{V}, \mathcal{V}_0) \rightarrow \text{GL}(\mathcal{V}/\mathcal{V}_0)$ defined by $\varphi_1(A) = A|_{\mathcal{V}_0}$, $\varphi_2(A)(v + \mathcal{V}_0) = Av + \mathcal{V}_0$. It is easy to see that $\rho(G) \subset \text{GL}(\mathcal{V}, \mathcal{V}_0)$ and $\rho_i(G) = \varphi_i(\rho(G))$, $i = 1, 2$.

(1) Let N be a normal cocompact algebraic subgroup of $\overline{\rho_i(G)}$, $i = 1$ or 2 . By [27, Cor. 4.6.5], $\varphi_i(\overline{\rho(G)})$ is an open (and Zariski dense) subgroup of $\overline{\varphi_i(\rho(G))} = \overline{\rho_i(G)}$. So $N \cap \varphi_i(\overline{\rho(G)})$ is cocompact in $\varphi_i(\overline{\rho(G)})$, and hence $\varphi_i^{-1}(N) \cap \overline{\rho(G)}$ is cocompact in $\overline{\rho(G)}$. Since $\overline{\rho(G)}$ is discompact, we have $\overline{\rho(G)} \subset \varphi_i^{-1}(N)$. Thus $\varphi_i(\overline{\rho(G)}) \subset N$. But $\varphi_i(\overline{\rho(G)})$ is Zariski dense in $\overline{\rho_i(G)}$. So $N = \overline{\rho_i(G)}$. Hence $\overline{\rho_i(G)}$ is discompact, i.e., G is ρ_i -discompact.

(2) Suppose that G is connected and ρ_i -discompact, $i = 1, 2$. Then Proposition 5.3 implies that all eigenvalues of $\rho_i(g)$ are real for every $g \in R(G)$. Hence all eigenvalues of $\rho(g)$ are real for every $g \in R(G)$. By Proposition 5.3, to prove that G is ρ -discompact, it suffices to show that any Levi subgroup L of $\rho(G)$ has no compact factors. Suppose on the contrary that L has a compact simple factor L' . Since $\varphi_i(L)$ is a Levi subgroup of $\varphi_i(\rho(G)) = \rho_i(G)$, Proposition 5.3 implies that $\varphi_i(L)$ has no compact factors. So $L' \subset \ker(\varphi_1) \cap \ker(\varphi_2)$. But $\ker(\varphi_1) \cap \ker(\varphi_2)$ is unipotent, a contradiction. This completes the proof. \square

Remark 5.1. Lemma 5.5(2) does not hold without the connectedness condition. For example, let $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \setminus \{0\}, a^2 = b^2 \right\}$, let ρ be the natural representation of G in $\mathcal{V} = \mathbb{R}^2$, and let $\mathcal{V}_0 = \mathbb{R} \times \{0\}$. Since G is a real algebraic group and is not Zariski connected, it is not ρ -discompact. But G is discompact with respect to the subrepresentation in \mathcal{V}_0 and the quotient representation in $\mathcal{V}/\mathcal{V}_0$.

Proof of Lemma 1.1. (1) Denote the quotient representation of G in \mathcal{V}/\mathcal{G} by $\rho_{\mathcal{V}/\mathcal{G}}$. By Lemma 5.5(2), it suffices to prove that G is discompact with respect to $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{V}/\mathcal{G}}$. Since the induced infinitesimal action of \mathfrak{g} on M is G -equivariant, $\rho_{\mathcal{G}}$ is equivalent to a quotient representation of the adjoint representation of G . By Corollary 5.4 and Lemma 5.5(1), G is $\rho_{\mathcal{G}}$ -discompact. On the other hand, the conditions in Lemma 1.1(1) imply that all eigenvalues of $\rho_{\mathcal{V}/\mathcal{G}}(g)$ are real for every $g \in R(G)$, and that any Levi subgroup of $\rho_{\mathcal{V}/\mathcal{G}}(G)$ has no compact factors. By Proposition 5.3, G is $\rho_{\mathcal{V}/\mathcal{G}}$ -discompact.

(2) Suppose that G is $\rho_{\mathcal{V}}$ -discompact and the G -action is faithful. By Lemma 5.5(1), G is $\rho_{\mathcal{G}}$ -discompact and $\rho_{\mathcal{V}/\mathcal{G}}$ -discompact. The faithfulness of the G -action

implies that the induced infinitesimal action of \mathfrak{g} on M is injective. So ρ_G is equivalent to the adjoint representation of G . Hence G is Ad-discompact. By Corollary 5.4, $R(G)$ is split solvable and $G/R(G)$ has no compact factors. On the other hand, since G is $\rho_{\mathcal{V}/\mathcal{G}}$ -discompact, Proposition 5.3 implies that all eigenvalues of $\rho_{\mathcal{V}/\mathcal{G}}(g)$ are real for every $g \in R(G)$. By [22, Cor. 1.29], $R(G)$ fixes a full flag of \mathcal{V}/\mathcal{G} , hence fixes a sequence of subspaces of \mathcal{V} with the required properties. \square

We conclude this section with the following result.

Proposition 5.6. *Let G be a Lie group, and let $\rho : G \rightarrow \mathrm{GL}(\mathcal{V})$ be a representation of G in a real vector space \mathcal{V} . Then there exists a unique maximal connected ρ -discompact Lie subgroup N of G . Moreover, N is a closed normal subgroup of G and G/N is locally isomorphic to a compact Lie group.*

Proof. Denote $D = R_d(\overline{\rho(G_0)})$. We prove that the group $N = \rho^{-1}(D)_0$ satisfies the requirement. We first show that N is ρ -discompact. Let L be a Levi subgroup of D . By Corollary 5.2, $R(D)$ is \mathbb{R} -split and L has no compact factors. By [27, Cor. 4.6.7], we have $L = (L, L) \subset (\overline{\rho(G_0)}, \overline{\rho(G_0)}) \subset \rho(G_0)$. Thus

$$(5.1) \quad L \subset (\rho(G) \cap D)_0 = \rho(\rho^{-1}(D))_0 = \rho(N).$$

Note that $\rho(N) \subset D_0$. So $D_0 = L \cdot R(D) = \rho(N) \cdot R(D)$. This implies that the quotient homomorphism $\varphi : D_0 \rightarrow D_0/R(D)$ maps $\rho(N)$ onto $D_0/R(D)$. So the connected solvable Lie subgroup $\varphi(R(\rho(N)))$ of $D_0/R(D)$ is normal, hence must be trivial. Thus $\rho(R(N)) = R(\rho(N)) \subset \ker(\varphi) = R(D)$. Since $R(D)$ is \mathbb{R} -split, all elements in $\rho(R(N))$ have only real eigenvalues. On the other hand, (5.1) also implies that L is a Levi subgroup of $\rho(N)$. Hence any Levi subgroup of $\rho(N)$ has no compact factors. By Proposition 5.3, N is ρ -discompact.

Now we prove that N contains every connected ρ -discompact Lie subgroup H of G . Since $\overline{\rho(H)} \subset \overline{\rho(G_0)}$ is discompact, we have $\overline{\rho(H)} \subset D$. So $H \subset \rho^{-1}(D)_0 = N$. Hence N is the unique maximal connected ρ -discompact Lie subgroup of G .

Obviously, N is closed in G . For any $g \in G$, $\rho(gNg^{-1}) = \rho(g)\overline{\rho(N)}\rho(g)^{-1}$ is discompact. So gNg^{-1} is ρ -discompact, and hence is contained in N . Thus N is normal in G . Consider the natural homomorphisms

$$G_0/N \rightarrow G_0/(\rho^{-1}(D) \cap G_0) \rightarrow \rho(G_0)/(D \cap \rho(G_0)) \rightarrow \overline{\rho(G_0)}/D$$

of Lie groups. The first is a covering homomorphism, the second is an isomorphism, and the third is injective. So $\mathcal{L}(G_0/N)$ is isomorphic to a subalgebra of $\mathcal{L}(\overline{\rho(G_0)}/D)$. Since $\overline{\rho(G_0)}/D$ is compact, $\mathcal{L}(G_0/N)$ is compact. Thus G/N is locally isomorphic to a compact Lie group. This completes the proof. \square

We refer to the group N in Proposition 5.6 as the *discompact radical of G with respect to ρ* (or the ρ -discompact radical of G), and denote it by $R_d^\rho(G)$.

Remark 5.2. (1) The quotient group $G/R_d^\rho(G)$ may be noncompact. Here is an example for the adjoint representation. Let $G = \mathbb{R} \ltimes \mathbb{C}^2$, where \mathbb{R} acts on \mathbb{C}^2 by $t.(z_1, z_2) = (e^{i\alpha t}z_1, e^{i\beta t}z_2)$, here $\alpha, \beta \in \mathbb{R}$ are linearly independent over \mathbb{Q} . It is easy to see that $R_d^{\mathrm{Ad}}(G) = \{0\} \times \mathbb{C}^2$. So $G/R_d^{\mathrm{Ad}}(G) \cong \mathbb{R}$.
 (2) Similar to the proof of Corollary 5.4, one can show that for any Lie group G , $R_d^{\mathrm{Ad}}(G)$ is the unique maximal member among all the connected normal

Lie subgroups N of G such that $R(N)$ is split solvable and $N/R(N)$ has no compact factors. Since we do not need this fact in this paper, we leave the details to the reader.

6. BOREL DENSITY THEOREM FOR ALGEBRAIC QUOTIENTS

The original form of the Borel density theorem asserts that any lattice in a real semisimple algebraic group without compact factors is Zariski dense ([7]). This was extended by several authors and played a crucial role in the ergodic theory of group actions (see [32, 38] and the references therein). One version of the theorem that we will need below is as follows ([32, Thms. 3.9 and 3.3]).

Theorem 6.1. *Let \mathbb{k} be a locally compact non-discrete field, \mathbf{G} be a \mathbb{k} -discompact \mathbb{k} -group, and \mathbf{K} be a \mathbb{k} -subgroup of \mathbf{G} . Suppose that there exists a $\mathbf{G}_{\mathbb{k}}$ -invariant finite Borel measure on $\mathbf{G}_{\mathbb{k}}/\mathbf{K}_{\mathbb{k}}$. Then $\mathbf{K} = \mathbf{G}$.*

In what follows we assume that $\text{char } \mathbb{k} = 0$, and consider certain action of $\mathbf{G}_{\mathbb{k}}$ on an algebraic quotient $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$, where \mathbf{G}, \mathbf{H} are \mathbb{k} -groups and \mathbf{V} is a \mathbb{k} -variety. More precisely, let $\mathbf{G} \times \mathbf{H}$ act \mathbb{k} -regularly on \mathbf{V} . Then $\mathbf{G}_{\mathbb{k}}$ acts naturally on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$. We endow $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ with the quotient topology and quotient Borel structure. Then the $\mathbf{G}_{\mathbb{k}}$ -action on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ is both continuous and measurable.

Recall that a Borel space is *countably separated* if there exists a countable family of Borel sets which separates points, and is *standard* if it is isomorphic to a Borel subset of a complete separable metric space. Obviously, standard Borel spaces are countably separated. We first list some basic properties of the topology and Borel structure on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$.

Lemma 6.2. (1) *For every $x \in \mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$, $\{x\}$ is locally closed.*
 (2) *$\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ is a standard Borel space.*
 (3) *The quotient Borel structure on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ coincides with the Borel structure generated by the quotient topology on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$.*

Proof. (1) follows from the local closedness of the $\mathbf{H}_{\mathbb{k}}$ -orbits in $\mathbf{V}_{\mathbb{k}}$. (2) follows from [15, Thm. 2.9] (see also [38, Thm. A.7]). (3) follows from [15, Thm. 2.6]. \square

Recall that a measurable action of an lcsc group G on a countably separated Borel space X is *tame* if the quotient Borel structure on X/G is countably separated. It is well-known the $(\mathbf{G}_{\mathbb{k}} \times \mathbf{H}_{\mathbb{k}})$ -action on $\mathbf{V}_{\mathbb{k}}$ is tame.

Lemma 6.3. (1) *The $\mathbf{G}_{\mathbb{k}}$ -action on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ is tame.*
 (2) *For every $x \in \mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$, there exists a \mathbb{k} -subgroup \mathbf{K} of \mathbf{G} such that $\mathbf{G}_{\mathbb{k}}(x)$ is an open subgroup of finite index in $\mathbf{K}_{\mathbb{k}}$.*

Proof. (1) Since $(\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}})/\mathbf{G}_{\mathbb{k}}$ is isomorphic to $\mathbf{V}_{\mathbb{k}}/(\mathbf{G}_{\mathbb{k}} \times \mathbf{H}_{\mathbb{k}})$ as Borel spaces, the tameness of the $\mathbf{G}_{\mathbb{k}}$ -action on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ follows from the tameness of the $(\mathbf{G}_{\mathbb{k}} \times \mathbf{H}_{\mathbb{k}})$ -action on $\mathbf{V}_{\mathbb{k}}$.

(2) Let $v \in x$, and denote $\mathbf{S} = (\mathbf{G} \times \mathbf{H})(v)$. Since $\text{char } \mathbb{k} = 0$, \mathbf{S} is a \mathbb{k} -subgroup of $\mathbf{G} \times \mathbf{H}$. Let $p_1 : \mathbf{G} \times \mathbf{H} \rightarrow \mathbf{G}$ be the projection. It is easy to see that $\mathbf{G}_{\mathbb{k}}(x) = p_1(\mathbf{S}_{\mathbb{k}})$. Let $\mathbf{K} = p_1(\mathbf{S})$. Then \mathbf{K} is a \mathbb{k} -subgroup of \mathbf{G} , and $p_1(\mathbf{S}_{\mathbb{k}})$ is an open subgroup of finite index in $\mathbf{K}_{\mathbb{k}}$ (see [29, p. 113, Cor. 1 and p. 319, Cor. 2]). Hence the result follows. \square

Now we prove the Borel density theorem for algebraic quotients (Theorem 1.8).

Theorem 6.4. *Let \mathbb{k} be a locally compact non-discrete field of characteristic 0, \mathbf{G} and \mathbf{H} be \mathbb{k} -groups such that \mathbf{G} is \mathbb{k} -discompact, and \mathbf{V} be a \mathbb{k} -variety such that $\mathbf{V}_{\mathbb{k}} \neq \emptyset$. Suppose that $\mathbf{G} \times \mathbf{H}$ acts \mathbb{k} -regularly on \mathbf{V} . Let μ be a finite Borel measure on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ such that either*

- (1) μ is $\mathbf{G}_{\mathbb{k}}$ -invariant, or
- (2) $\mathbb{k} = \mathbb{R}$ and μ is invariant under a Zariski dense Lie subgroup G of $\mathbf{G}_{\mathbb{R}}$.

Then for μ -a.e. $x \in \mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$, $\mathbf{G}_{\mathbb{k}}(x)$ is an open subgroup of finite index in $\mathbf{G}_{\mathbb{k}}$.

Proof. If $\mathbb{k} \neq \mathbb{R}$, we let $G = \mathbf{G}_{\mathbb{k}}$. Since $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ is a standard Borel G -space and G is lcsc (if $\mathbb{k} = \mathbb{R}$ we consider the Lie group topology on G), by [34, Thm. 4.4], the G -invariant measure μ admits an ergodic decomposition. This means that there exists a map $x \mapsto \beta_x$ from $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ to the space of G -invariant ergodic probability measures on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ such that for any Borel set $E \subset \mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$, the function $x \mapsto \beta_x(E)$ is measurable and we have $\mu(E) = \int_X \beta_x(E) d\mu(x)$. So we may assume without loss of generality that μ is G -ergodic. Since the $\mathbf{G}_{\mathbb{k}}$ -action on $\mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$ is tame, by [32, Lem. 2.2], μ is supported on a single $\mathbf{G}_{\mathbb{k}}$ -orbit, say $\mathcal{O} \subset \mathbf{V}_{\mathbb{k}}/\mathbf{H}_{\mathbb{k}}$. It suffices to prove that for every $x \in \mathcal{O}$, $\mathbf{G}_{\mathbb{k}}(x)$ is an open subgroup of finite index in $\mathbf{G}_{\mathbb{k}}$.

Let $x \in \mathcal{O}$. By Lemma 6.3(2), there exists a \mathbb{k} -subgroup \mathbf{K} of \mathbf{G} such that $\mathbf{G}_{\mathbb{k}}(x)$ is an open subgroup of finite index in $\mathbf{K}_{\mathbb{k}}$. In particular, $\mathbf{G}_{\mathbb{k}}(x)$ is closed in $\mathbf{G}_{\mathbb{k}}$. So $\mathbf{G}_{\mathbb{k}}/\mathbf{G}_{\mathbb{k}}(x)$ is a standard Borel space. By [38, Thm. A.4], $\mathbf{G}_{\mathbb{k}}/\mathbf{G}_{\mathbb{k}}(x)$ is isomorphic to \mathcal{O} as Borel $\mathbf{G}_{\mathbb{k}}$ -spaces. So we may view μ as a G -invariant finite Borel measure on $\mathbf{G}_{\mathbb{k}}/\mathbf{G}_{\mathbb{k}}(x)$. Let $p : \mathbf{G}_{\mathbb{k}}/\mathbf{G}_{\mathbb{k}}(x) \rightarrow \mathbf{G}_{\mathbb{k}}/\mathbf{K}_{\mathbb{k}}$ be the natural projection. Then $p_*(\mu)$ is a G -invariant finite Borel measure on $\mathbf{G}_{\mathbb{k}}/\mathbf{K}_{\mathbb{k}}$. We claim that $p_*(\mu)$ is $\mathbf{G}_{\mathbb{k}}$ -invariant. Indeed, if $\mathbb{k} \neq \mathbb{R}$, this has been assumed as a condition. If $\mathbb{k} = \mathbb{R}$, since G is Zariski dense in $\mathbf{G}_{\mathbb{R}}$, the $\mathbf{G}_{\mathbb{R}}$ -invariance of $p_*(\mu)$ follows from [13, Cor. 2.6]. Now by Theorem 6.1, we have $\mathbf{K} = \mathbf{G}$. Thus $\mathbf{G}_{\mathbb{k}}(x)$ is an open subgroup of finite index in $\mathbf{G}_{\mathbb{k}}$. This completes the proof. \square

Remark 6.1. Under the conditions of Theorem 6.4, even if $\mathbb{k} = \mathbb{R}$, $\mathbf{G}_{\mathbb{R}}$ may have no fixed point in $\mathbf{V}_{\mathbb{R}}/\mathbf{H}_{\mathbb{R}}$. For example, let $\mathbf{G} = \mathbf{H} = \mathbf{V} = \mathbb{C} \setminus \{0\}$ be the 1-dimensional \mathbb{R} -split torus, and let $\mathbf{G} \times \mathbf{H}$ act on \mathbf{V} by $(g, h).v = gh^2v$. Then $\mathbf{V}_{\mathbb{R}}/\mathbf{H}_{\mathbb{R}}$ consists of two points, each of them has stabilizer $(0, +\infty)$, which is not an algebraic subgroup of $\mathbf{G}_{\mathbb{R}}$. Recall that by Rosenlicht's theorem [31], $\mathbf{V}_{\mathbb{R}}/\mathbf{H}_{\mathbb{R}}$ is a finite disjoint union $\bigcup_i V_i$ of real algebraic varieties. The above example shows that even if each V_i can be chosen to be $\mathbf{G}_{\mathbb{R}}$ -invariant, the $\mathbf{G}_{\mathbb{R}}$ -action on V_i may fail to be algebraic (for otherwise the stabilizers would be algebraic subgroups of $\mathbf{G}_{\mathbb{R}}$).

For the convenience of later applications, we propose the following corollary of Theorem 6.4.

Corollary 6.5. *Let G be a connected Lie group, $\rho : G \rightarrow \mathrm{GL}(\mathcal{V})$ be a representation of G in a real vector space \mathcal{V} such that G is ρ -discompact, H be a real algebraic group, and V be a real algebraic variety. Suppose that $\mathrm{GL}(\mathcal{V}) \times H$ acts regularly on V , and that the induced G -action on V/H preserves a finite Borel measure μ on V/H . Then $(V/H)^G$ is a μ -conull G -saturated constructible subset of V/H .*

Proof. Theorem 6.4 implies that μ -a.e. point in V/H is fixed by some open subgroup of $\overline{\rho(G)}$. But an open subgroup of $\overline{\rho(G)}$ must contain $\rho(G)$. So $(V/H)^G$ is μ -conull. By Lemmas 6.2(1) and 2.5, for every $x \in (V/H)^G$, $\{x\}$ is G -saturated. So Lemma 2.4 implies that $(V/H)^G$ is G -saturated. The assertion that $(V/H)^G$ is

constructible follows from a standard inductive argument where we apply Lemma 2.3 to the set of nonsingular points in V . \square

7. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. For convenience, we restate the theorem as follows.

Theorem 7.1. *Let M be a connected n -dimensional C^ε manifold with an i -rigid C^ε A-structure σ of order r and type V , where $i \geq r > 0$, and let a connected Lie group G act on M by C^ε isometries. Let μ be a finite Borel measure on M , \mathcal{V} be a G -invariant subspace of $\text{Kill}(M)$, and \mathcal{W} be a Γ -invariant subspace of $\text{Kill}(\widetilde{M})^\mathcal{G}$, where $\Gamma = \pi_1(M)$. Suppose that G is ρ_V -discompact and $\ker(\rho_V) \cdot G_\mu = G$. Then there exists a G -saturated constructible set $M' \subset M_{\text{reg}}$ with $\mu(M_{\text{reg}} \setminus M') = 0$ such that for every $x \in M'$, we have*

- (1) *the $\text{Hull}_\Gamma^\mathcal{V}(\text{Iso}^{\text{germ}}(M)^\mathcal{V})$ -orbit of x contains a G -invariant open dense subset of \overline{Gx} , and*
- (2) *$T_x Gx \subset \text{ev}_x(\text{Hull}_\Gamma^\mathcal{W}(\text{Kill}_x^{\text{germ}}(M)^\mathcal{V}))$.*

The proof of the theorem generalizes ideas in the proofs of Gromov's centralizer theorem in [21, 41]. We make use of two geometric structures associated with the i -jets of Killing fields in \mathcal{V} and \mathcal{W} , respectively. Let $k > i$ be an integer. We first introduce a notation. Let N_1 and N_2 be n -dimensional C^ε manifolds, and let $x \in N_1$, $y \in N_2$. If f is a local C^ε diffeomorphism from N_1 to N_2 defined around x sending x to y and $\xi \in \text{Vect}_x^{\text{germ}}(N_1)$, then we denote $(j_x^k f)_*(j_x^i \xi) = j_y^i df(\xi)$. Since $k > i$, $(j_x^k f)_*(j_x^i \xi)$ depends only on $j_x^k f$ and $j_x^i \xi$. This defines a linear isomorphism $(j_x^k f)_* : \text{Vect}_x^i(N_1) \rightarrow \text{Vect}_y^i(N_2)$.

Now we define the geometric structure induced by \mathcal{V} . This is motivated by [21, Sect. 5.1]. Let $L(\mathcal{V}, \text{Vect}_0^i(\mathbb{R}^n))$ denote the space of linear maps from \mathcal{V} to $\text{Vect}_0^i(\mathbb{R}^n)$. Then $\text{GL}^k(n) \times \text{GL}(\mathcal{V})$ acts regularly on $L(\mathcal{V}, \text{Vect}_0^i(\mathbb{R}^n))$ by

$$(\alpha, A) \cdot \ell = \alpha_* \circ \ell \circ A^{-1}, \quad \alpha \in \text{GL}^k(n), A \in \text{GL}(\mathcal{V}), \ell \in L(\mathcal{V}, \text{Vect}_0^i(\mathbb{R}^n)).$$

We define a map

$$\tau_{\mathcal{V}, k, i} : F^k(M) \rightarrow L(\mathcal{V}, \text{Vect}_0^i(\mathbb{R}^n))$$

by

$$\tau_{\mathcal{V}, k, i}(\beta)(v) = \beta_*^{-1}(j_x^i v), \quad x \in M, \beta \in F^k(M)_x, v \in \mathcal{V}.$$

Note that G acts on $F^k(M)$ by $g \cdot \beta = j_x^k g \circ \beta$, where $\beta \in F^k(M)_x$.

Lemma 7.2. *The map $\tau_{\mathcal{V}, k, i}$ is a C^ε A-structure, and is G -equivariant with respect to ρ_V .*

Proof. It suffices to check the $\text{GL}^k(n)$ -equivariance and G -equivariance of $\tau_{\mathcal{V}, k, i}$. Let $\alpha \in \text{GL}^k(n)$, $g \in G$, $x \in M$, $\beta \in F^k(M)_x$. Then for $v \in \mathcal{V}$ we have

$$\tau_{\mathcal{V}, k, i}(\beta \circ \alpha^{-1})(v) = (\alpha \circ \beta^{-1})_*(j_x^i v) = (\alpha_* \circ \tau_{\mathcal{V}, k, i}(\beta))(v) = (\alpha \cdot \tau_{\mathcal{V}, k, i}(\beta))(v)$$

and

$$\begin{aligned} \tau_{\mathcal{V}, k, i}(g \cdot \beta)(v) &= \tau_{\mathcal{V}, k, i}(j_x^k g \circ \beta)(v) = (\beta^{-1} \circ j_{gx}^k(g^{-1}))_*(j_{gx}^i v) \\ &= \beta_*^{-1}(j_x^i dg^{-1}(v)) = \beta_*^{-1}(j_x^i \rho_V(g^{-1})v) \\ &= (\tau_{\mathcal{V}, k, i}(\beta) \circ \rho_V(g)^{-1})(v) = (\rho_V(g) \cdot \tau_{\mathcal{V}, k, i}(\beta))(v). \end{aligned}$$

Hence $\tau_{\mathcal{V}, k, i}(\beta \circ \alpha^{-1}) = \alpha \cdot \tau_{\mathcal{V}, k, i}(\beta)$ and $\tau_{\mathcal{V}, k, i}(g \cdot \beta) = \rho_V(g) \cdot \tau_{\mathcal{V}, k, i}(\beta)$. \square

Next we define the geometric structure induced by \mathcal{W} . By applying the above construction to \widetilde{M} , we get a C^ε A-structure $\tau_{\mathcal{W},k,i} : F^k(\widetilde{M}) \rightarrow L(\mathcal{W}, \text{Vect}_0^i(\mathbb{R}^n))$. An argument similar to the proof of the second assertion of Lemma 7.2 shows that $\tau_{\mathcal{W},k,i}$ is Γ -equivariant with respect to the representation $\rho_{\mathcal{W}} : \Gamma \rightarrow \text{GL}(\mathcal{W})$. The map $F^k(\widetilde{M}) \rightarrow F^k(M)$ sending $\tilde{\beta} \in F^k(\widetilde{M})_{\tilde{x}}$ to $j_{\tilde{x}}^k \pi \circ \tilde{\beta}$ induces an identification $F^k(\widetilde{M})/\Gamma \cong F^k(M)$. Thus $\tau_{\mathcal{W},k,i}$ induces a continuous map

$$v_{\mathcal{W},k,i} : F^k(M) \rightarrow L(\mathcal{W}, \text{Vect}_0^i(\mathbb{R}^n)) / \overline{\rho_{\mathcal{W}}(\Gamma)},$$

where the algebraic quotient $L(\mathcal{W}, \text{Vect}_0^i(\mathbb{R}^n)) / \overline{\rho_{\mathcal{W}}(\Gamma)}$ is endowed with the quotient topology. Writing explicitly, we have

$$v_{\mathcal{W},k,i}(\beta) = \left(\overline{\rho_{\mathcal{W}}(\Gamma)} \right) (\tau_{\mathcal{W},k,i}((j_{\tilde{x}}^k \pi)^{-1} \circ \beta)), \quad x \in M, \beta \in F^k(M)_x, \tilde{x} \in \pi^{-1}(x),$$

where

$$\tau_{\mathcal{W},k,i}((j_{\tilde{x}}^k \pi)^{-1} \circ \beta)(w) = \beta_*^{-1}(d\pi(j_{\tilde{x}}^i w)), \quad w \in \mathcal{W}.$$

Note that $\text{GL}^k(n)$ acts continuously on $L(\mathcal{W}, \text{Vect}_0^i(\mathbb{R}^n)) / \overline{\rho_{\mathcal{W}}(\Gamma)}$.

Lemma 7.3. *The map $v_{\mathcal{W},k,i}$ is a continuous G -invariant geometric structure.*

Proof. The $\text{GL}^k(n)$ -equivariance of $v_{\mathcal{W},k,i}$ follows from that of $\tau_{\mathcal{W},k,i}$. To prove the G -invariance, we consider the induced action of \tilde{G} on \widetilde{M} . Since $\mathcal{W} \subset \text{Kill}(\widetilde{M})^{\mathcal{G}}$, \mathcal{W} is \tilde{G} -invariant and the representation $\tilde{G} \rightarrow \text{GL}(\mathcal{W})$ is trivial. An argument similar to the proof of Lemma 7.2 shows that $\tau_{\mathcal{W},k,i}$ is \tilde{G} -invariant. By passing to the quotients, we obtain the G -invariance of $v_{\mathcal{W},k,i}$. \square

Now we are prepared to prove Theorem 7.1. We first prove assertion (1).

Proof of Theorem 7.1(1). Let $k > i$ be such that every point in M_{reg} is k -regular. Consider the $(k-r)$ -th prolongation $\sigma^{k-r} : F^k(M) \rightarrow J_n^{k-r}(V)$ of σ and the geometric structures $\tau_{\mathcal{V},k,i}$ and $v_{\mathcal{W},k,i}$ constructed above. Since the G -action on M is isometric, σ^{k-r} is G -invariant. By Lemmas 7.2 and 7.3, $\tau_{\mathcal{V},k,i}$ is G -equivariant with respect to $\rho_{\mathcal{V}}$, and $v_{\mathcal{W},k,i}$ is G -invariant. Consider the regular action of $\text{GL}^k(n) \times \text{GL}(\mathcal{V}) \times \text{GL}(\mathcal{W})$ on

$$W = J_n^{k-r}(V) \times L(\mathcal{V}, \text{Vect}_0^i(\mathbb{R}^n)) \times L(\mathcal{W}, \text{Vect}_0^i(\mathbb{R}^n))$$

defined by

$$(\alpha, A_1, A_2).(\zeta, \ell_1, \ell_2) = (\alpha.\zeta, \alpha_* \circ \ell_1 \circ A_1^{-1}, \alpha_* \circ \ell_2 \circ A_2^{-1}),$$

where $(\alpha, A_1, A_2) \in \text{GL}^k(n) \times \text{GL}(\mathcal{V}) \times \text{GL}(\mathcal{W})$ and $(\zeta, \ell_1, \ell_2) \in W$. Then there is a natural identification

$$W / \overline{\rho_{\mathcal{W}}(\Gamma)} = J_n^{k-r}(V) \times L(\mathcal{V}, \text{Vect}_0^i(\mathbb{R}^n)) \times \left(L(\mathcal{W}, \text{Vect}_0^i(\mathbb{R}^n)) / \overline{\rho_{\mathcal{W}}(\Gamma)} \right).$$

The continuous geometric structure

$$\sigma^{k-r} \times \tau_{\mathcal{V},k,i} \times v_{\mathcal{W},k,i} : F^k(M) \rightarrow W / \overline{\rho_{\mathcal{W}}(\Gamma)}$$

is G -equivariant with respect to $\rho_{\mathcal{V}}$. So its continuous Gauss map

$$\theta : M \rightarrow W / \left(\text{GL}^k(n) \times \overline{\rho_{\mathcal{W}}(\Gamma)} \right)$$

is also G -equivariant with respect to $\rho_{\mathcal{V}}$. Since θ is measurable, it induces a finite Borel measure $\theta_*(\mu)$ on $W / \left(\mathrm{GL}^k(n) \times \overline{\rho_{\mathcal{W}}(\Gamma)} \right)$. The condition $\ker(\rho_{\mathcal{V}}) \cdot G_{\mu} = G$ implies that $\theta_*(\mu)$ is G -invariant. Since G is $\rho_{\mathcal{V}}$ -discompact, by Corollary 6.5, the set

$$F = \left(W / \left(\mathrm{GL}^k(n) \times \overline{\rho_{\mathcal{W}}(\Gamma)} \right) \right)^G$$

is G -saturated, constructible, and $\theta_*(\mu)$ -conull. Let

$$M' = \theta^{-1}(F) \cap M_{\mathrm{reg}}.$$

Obviously, $\theta^{-1}(F)$ is constructible and μ -conull. By Lemma 2.6, $\theta^{-1}(F)$ is G -saturated. So M' is G -saturated and constructible, and satisfies $\mu(M_{\mathrm{reg}} \setminus M') = 0$. We prove that every $x \in M'$ satisfies the requirement of Theorem 7.1(1).

Let $p = \theta(x)$. Then p is fixed by G . By Lemma 6.2(1), $\{p\}$ is locally closed. So $\theta^{-1}(p)$, and hence $\theta^{-1}(p) \cap M_{\mathrm{reg}}$, is G -invariant and locally closed. Let C_x be the connected component of $\theta^{-1}(p) \cap M_{\mathrm{reg}}$ containing x . We claim that C_x is also G -invariant and locally closed. Indeed, since C_x is closed in $\theta^{-1}(p) \cap M_{\mathrm{reg}}$, it is locally closed. To see the G -invariance of C_x , let $y \in C_x$. Since Gy and C_x are connected and $Gy \cap C_x \neq \emptyset$, $Gy \cup C_x$ is also connected. But $Gy \cup C_x$ is contained in $\theta^{-1}(p) \cap M_{\mathrm{reg}}$ and C_x is a connected component of $\theta^{-1}(p) \cap M_{\mathrm{reg}}$. So we must have $Gy \cup C_x = C_x$. Thus $Gy \subset C_x$, and hence C_x is G -invariant. Let $S_x = C_x \cap \overline{Gx}$, which is obviously G -invariant. We claim that S_x is open dense in \overline{Gx} . Indeed, since C_x is open in $\overline{C_x}$, S_x is open in $\overline{C_x} \cap \overline{Gx} = \overline{Gx}$. On the other hand, since $Gx \subset S_x$, S_x is dense in \overline{Gx} . Thus it suffices to prove that the $\mathrm{Hull}_{\Gamma}^{\mathcal{W}}(\mathrm{Iso}^{\mathrm{germ}}(M)^{\mathcal{V}})$ -orbit of x contains S_x .

We prove the stronger assertion that the $\mathrm{Hull}_{\Gamma}^{\mathcal{W}}(\mathrm{Iso}^{\mathrm{germ}}(M)^{\mathcal{V}})$ -orbit of x contains C_x . Since C_x is connected, we need only to show that for every $\mathrm{Hull}_{\Gamma}^{\mathcal{W}}(\mathrm{Iso}^{\mathrm{germ}}(M)^{\mathcal{V}})$ -orbit O in M , $O \cap C_x$ is open in C_x . If $O \cap C_x = \emptyset$, there is nothing to prove. Otherwise, for any $y \in O \cap C_x$, since $y \in M_{\mathrm{reg}}$, there exists an open neighborhood U_y of y in M such that $\mathrm{Iso}_{y,z}^{\mathrm{germ}}(M) \rightarrow \mathrm{Iso}_{y,z}^k(M)$ is surjective for every $z \in U_y$. We will prove that $U_y \cap C_x \subset O$. If this is true, then the open neighborhood $U_y \cap C_x$ of y in C_x will be contained in $O \cap C_x$, and hence $O \cap C_x$ will be open in C_x .

It remains to prove that $U_y \cap C_x \subset O$. Let $z \in U_y \cap C_x$. Choose $\beta_y \in F^k(M)_y$ and $\beta_z \in F^k(M)_z$. Since $y, z \in C_x$, we have $\theta(z) = \theta(y)$. So there exists $\alpha \in \mathrm{GL}^k(n)$ such that

$$\alpha \cdot (\sigma^{k-r} \times \tau_{\mathcal{V},k,i} \times v_{\mathcal{W},k,i})(\beta_y) = (\sigma^{k-r} \times \tau_{\mathcal{V},k,i} \times v_{\mathcal{W},k,i})(\beta_z).$$

Hence we have

$$(7.1) \quad \alpha \cdot \sigma^{k-r}(\beta_y) = \sigma^{k-r}(\beta_z),$$

$$(7.2) \quad \alpha \cdot \tau_{\mathcal{V},k,i}(\beta_y) = \tau_{\mathcal{V},k,i}(\beta_z),$$

$$(7.3) \quad \alpha \cdot v_{\mathcal{W},k,i}(\beta_y) = v_{\mathcal{W},k,i}(\beta_z).$$

Equation (7.1) implies that

$$\sigma^{k-r}((\beta_z \circ \alpha \circ \beta_y^{-1}) \circ \beta_y) = \alpha^{-1} \cdot \sigma^{k-r}(\beta_z) = \sigma^{k-r}(\beta_y).$$

So $\beta_z \circ \alpha \circ \beta_y^{-1} \in \mathrm{Iso}_{y,z}^k(M)$. Since $\mathrm{Iso}_{y,z}^{\mathrm{germ}}(M) \rightarrow \mathrm{Iso}_{y,z}^k(M)$ is surjective, there exists $\varphi \in \mathrm{Iso}_{y,z}^{\mathrm{germ}}(M)$ such that $j_y^k \varphi = \beta_z \circ \alpha \circ \beta_y^{-1}$. By (7.2), for any $v \in \mathcal{V}$ we

have

$$\begin{aligned} j_z^i d\varphi(\langle v \rangle_y) &= (j_y^k \varphi)_*(j_y^i v) = (\beta_z \circ \alpha \circ \beta_y^{-1})_*(j_y^i v) = (\beta_z)_*(\alpha \cdot \tau_{\mathcal{V}, k, i}(\beta_y)(v)) \\ &= (\beta_z)_*(\tau_{\mathcal{V}, k, i}(\beta_z)(v)) = (\beta_z)_*((\beta_z^{-1})_*(j_z^i v)) = j_z^i(\langle v \rangle_z). \end{aligned}$$

Since $\text{Kill}_z^{\text{germ}}(M) \rightarrow \text{Vect}_z^i(M)$ is injective, we have $d\varphi(\langle v \rangle_y) = \langle v \rangle_z$ for all $v \in \mathcal{V}$. So $\varphi \in \text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$. Let $\tilde{y} \in \pi^{-1}(y)$, $\tilde{z} \in \pi^{-1}(z)$, $\beta_{\tilde{y}} = (j_{\tilde{y}}^k \pi)^{-1} \circ \beta_y$, $\beta_{\tilde{z}} = (j_{\tilde{z}}^k \pi)^{-1} \circ \beta_z$. By (7.3), there exists $A \in \overline{\rho_{\mathcal{W}}(\Gamma)}$ such that

$$\alpha \cdot \tau_{\mathcal{W}, k, i}(\beta_{\tilde{y}}) = \tau_{\mathcal{W}, k, i}(\beta_{\tilde{z}}) \circ A.$$

Let $\lambda_{\tilde{y}}$ and $\lambda_{\tilde{z}}$ be the homomorphisms defined in (4.1). Then for any $w \in \mathcal{W}$ we have

$$\begin{aligned} j_z^i d\varphi(\lambda_{\tilde{y}}(w)) &= j_z^i d\varphi(d\pi\langle w \rangle_{\tilde{y}}) = (j_y^k \varphi \circ j_{\tilde{y}}^k \pi)_*(j_{\tilde{y}}^i w) = (\beta_z \circ \alpha \circ \beta_y^{-1} \circ j_{\tilde{y}}^k \pi)_*(j_{\tilde{y}}^i w) \\ &= (\beta_z \circ \alpha \circ \beta_{\tilde{y}}^{-1})_*(j_{\tilde{y}}^i w) = (\beta_z)_*(\alpha \cdot \tau_{\mathcal{W}, k, i}(\beta_{\tilde{y}})(w)) \\ &= (\beta_z)_*(\tau_{\mathcal{W}, k, i}(\beta_{\tilde{z}})(Aw)) = (\beta_z)_*((\beta_{\tilde{z}}^{-1})_*(j_{\tilde{z}}^i Aw)) \\ &= (j_{\tilde{z}}^k \pi)_*(j_{\tilde{z}}^i Aw) = j_z^i d\pi(\langle Aw \rangle_{\tilde{z}}) = j_z^i \lambda_{\tilde{z}}(Aw). \end{aligned}$$

By the injectivity of $\text{Kill}_z^{\text{germ}}(M) \rightarrow \text{Vect}_z^i(M)$, we have $d\varphi(\lambda_{\tilde{y}}(w)) = \lambda_{\tilde{z}}(Aw)$ for all $w \in \mathcal{W}$. So $\varphi \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Iso}^{\text{germ}}(M)^{\mathcal{V}})$. Thus z lies in the $\text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Iso}^{\text{germ}}(M)^{\mathcal{V}})$ -orbit O of y . This proves that $U_y \cap C_x \subset O$, and hence completes the proof of Theorem 7.1(1). \square

Now we deduce the second assertion of the theorem from the first one.

Proof of Theorem 7.1(2). Note that for $x \in M$, a G -invariant open dense subset of \overline{Gx} must contain Gx . So by Theorem 7.1(1), it suffices to prove that for $x \in M$, if the condition

$$(7.4) \quad Gx \text{ is contained in the } \text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Iso}^{\text{germ}}(M)^{\mathcal{V}})\text{-orbit of } x$$

holds, then $T_x Gx \subset \text{ev}_x(\text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}}))$.

Let \mathcal{V}_x be the image of \mathcal{V} under the injection $\text{Kill}(M) \rightarrow \text{Kill}_x^{\text{germ}}(M)$. For $v \in \mathcal{V}$, we denote $\lambda_x(v) = \langle v \rangle_x$. Then we have an isomorphism $\lambda_x : \mathcal{V} \rightarrow \mathcal{V}_x$. Let

$$\mathcal{A}_x = \{\varphi \in \text{Iso}_x^{\text{germ}}(M) \mid d\varphi(\mathcal{V}_x) = \mathcal{V}_x\}.$$

Then \mathcal{A}_x is a closed subgroup of $\text{Iso}_x^{\text{germ}}(M)$. For $\varphi \in \mathcal{A}_x$, we denote $\rho(\varphi) = \lambda_x^{-1} \circ d\varphi \circ \lambda_x$. Then $\rho : \mathcal{A}_x \rightarrow \text{GL}(\mathcal{V})$ is a representation. We first prove that if (7.4) holds, then

$$(7.5) \quad \rho_{\mathcal{V}}(G) \subset \rho(\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_x)).$$

Let $g \in G$. Then (7.4) implies that there exists $\psi \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Iso}^{\text{germ}}(M)^{\mathcal{V}})$ with source gx and target x . Let $\varphi = \psi \circ \langle g \rangle_x \in \text{Iso}_x^{\text{germ}}(M)$. We prove (7.5) by showing that $\varphi \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_x)$ and $\rho(\varphi) = \rho_{\mathcal{V}}(g)$. Since ψ centralizes \mathcal{V} , for any $v \in \mathcal{V}$ we have

$$\begin{aligned} d\varphi \circ \lambda_x(v) &= d(\psi \circ \langle g \rangle_x)(\langle v \rangle_x) = d\psi(\langle dg(v) \rangle_{gx}) \\ &= \langle dg(v) \rangle_x = \lambda_x \circ \rho_{\mathcal{V}}(g)(v). \end{aligned}$$

So $\varphi \in \mathcal{A}_x$ and $\rho(\varphi) = \rho_{\mathcal{V}}(g)$. Consider the induced action of \tilde{G} on \tilde{M} . Let \tilde{g} be an element in the preimage of g under the covering homomorphism $\tilde{G} \rightarrow G$, and let $\tilde{x} \in \pi^{-1}(x)$. Then $\pi(\tilde{g}\tilde{x}) = gx$. By the choice of ψ , there exists $A \in \overline{\rho_{\mathcal{W}}(\Gamma)}$ such

that $d\psi(d\pi(\langle w \rangle_{\tilde{g}\tilde{x}})) = d\pi(\langle Aw \rangle_{\tilde{x}})$ for all $w \in \mathcal{W}$. Since $\mathcal{W} \subset \text{Kill}(\widetilde{M})^{\mathcal{G}}$, we have $d\tilde{g}(w) = w$ for all $w \in \mathcal{W}$. Thus

$$\begin{aligned} d\varphi(d\pi(\langle w \rangle_{\tilde{x}})) &= d\psi(d(g \circ \pi)(\langle w \rangle_{\tilde{x}})) = d\psi(d(\pi \circ \tilde{g})(\langle w \rangle_{\tilde{x}})) \\ &= d\psi(d\pi(\langle d\tilde{g}(w) \rangle_{\tilde{g}\tilde{x}})) = d\psi(d\pi(\langle w \rangle_{\tilde{g}\tilde{x}})) = d\pi(\langle Aw \rangle_{\tilde{x}}) \end{aligned}$$

for all $w \in \mathcal{W}$. Hence $\varphi \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_x)$. This proves (7.5). Note that (7.5) and Lemma 4.5 imply that

$$(7.6) \quad d\rho_{\mathcal{V}}(\mathfrak{g}) = \mathcal{L}(\rho_{\mathcal{V}}(G)) \subset \mathcal{L}(\rho(\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{A}_x))) = d\rho(\text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{L}(\mathcal{A}_x))).$$

Note also that by taking Lie derivatives, we have

$$\mathcal{L}(\mathcal{A}_x) = \{\xi \in \text{Kill}_x^{\text{germ}}(M)_0 \mid [\xi, \mathcal{V}_x] \subset \mathcal{V}_x\},$$

and

$$d\rho_{\mathcal{V}}(X)(v) = [\iota(X), v], \quad d\rho(\xi)(v) = \lambda_x^{-1}([\xi, \langle v \rangle_x]), \quad X \in \mathfrak{g}, \xi \in \mathcal{L}(\mathcal{A}_x), v \in \mathcal{V}.$$

Now we prove that if (7.6) holds, then $T_x Gx \subset \text{ev}_x(\text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}}))$. Let $X \in \mathfrak{g}$. By (7.6), there exists $\xi \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{L}(\mathcal{A}_x))$ such that $d\rho_{\mathcal{V}}(X) = d\rho(\xi)$. This means that

$$[\langle \iota(X) \rangle_x, \langle v \rangle_x] = \langle [\iota(X), v] \rangle_x = [\xi, \langle v \rangle_x]$$

for all $v \in \mathcal{V}$. Let $\eta = \langle \iota(X) \rangle_x - \xi$. Then $\eta \in \text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}}$. Moreover, since $\xi \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathcal{L}(\mathcal{A}_x))$ and $\mathcal{W} \subset \text{Kill}(\widetilde{M})^{\mathcal{G}}$, there exists $B \in \mathcal{L}(\overline{\rho_{\mathcal{W}}(\Gamma)})$ such that

$$[\eta, d\pi(\langle w \rangle_{\tilde{x}})] = -[\xi, d\pi(\langle w \rangle_{\tilde{x}})] = -d\pi(\langle Bw \rangle_{\tilde{x}})$$

for all $w \in \mathcal{W}$. Thus $\eta \in \text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}})$. Note that $\text{ev}_x(\xi) = 0$. So $\text{ev}_x(\eta) = \iota_x(X)$. Hence

$$T_x Gx = \iota_x(\mathfrak{g}) \subset \text{ev}_x(\text{Hull}_{\Gamma}^{\mathcal{W}}(\text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}})).$$

This proves Theorem 7.1(2). \square

8. GENERALIZATIONS OF GROMOV'S THEOREMS

As explained in Section 1, Theorem 1.2 unifies Gromov's centralizer, representation, and open dense orbit theorems. In this section we provide the details. We first prove Theorem 1.3, which generalizes the centralizer and representation theorems. Throughout this section we denote $\Gamma = \pi_1(M)$.

Proof of Theorem 1.3. Let $\mathcal{W} = \text{Kill}(\widetilde{M})^{\mathcal{V}}$, which is obviously Γ -invariant and contained in $\text{Kill}(\widetilde{M})^{\mathcal{G}}$. For $x \in M$, let \mathcal{G}_x be the image of \mathcal{G} under the injection $\text{Kill}(M) \rightarrow \text{Kill}_x^{\text{germ}}(M)$, and denote

$$\mathfrak{a}_x = \text{Kill}_x^{\text{germ}}(M)^{\mathcal{V}}, \quad \mathfrak{h}_x = \text{Hull}_{\Gamma}^{\mathcal{W}}(\mathfrak{a}_x), \quad \mathfrak{f}_x = \mathcal{G}_x + \mathfrak{h}_x.$$

Then $[\mathcal{G}_x, \mathfrak{h}_x] = 0$. By Theorem 1.2(2), there exists a G -saturated constructible set $M' \subset M_{\text{reg}}$ with $\mu(M_{\text{reg}} \setminus M') = 0$ such that

$$(8.1) \quad T_x Gx \subset \text{ev}_x(\mathfrak{h}_x), \quad x \in M'.$$

We prove that every $x \in M'$ satisfies the requirements of Theorem 1.3.

Firstly, since $\mathfrak{h}_x < \mathfrak{a}_x$, from (8.1) we obtain $T_x Gx \subset \text{ev}_x(\mathfrak{a}_x)$. On the other hand, since $\text{ev}_x(\mathcal{G}_x) = T_x Gx$, (8.1) implies that \mathfrak{f}_x is equal to the sum of \mathfrak{h}_x and the kernel of the evaluation map $\text{ev}_x : \mathfrak{f}_x \rightarrow T_x M$, which is a subalgebra of \mathfrak{f}_x . By Lemma 2.10, there exist $\mathfrak{z}_x < \mathfrak{h}_x$ and a Lie algebra structure on $T_x Gx$ such that

$\text{ev}_x(\mathfrak{z}_x) = T_x Gx$, and such that $\text{ev}_x : \mathcal{G}_x \rightarrow T_x Gx$ and $-\text{ev}_x : \mathfrak{z}_x \rightarrow T_x Gx$ are Lie algebra homomorphisms. In particular, we have $T_x Gx \prec \mathfrak{h}_x$. This also implies that $\iota_x = \text{ev}_x \circ \langle \cdot \rangle_x \circ \iota : \mathfrak{g} \rightarrow T_x Gx$ is a homomorphism. Thus $\mathfrak{g}(x) = \ker(\iota_x) \triangleleft \mathfrak{g}$, and we have

$$(8.2) \quad \mathfrak{g}/\mathfrak{g}(x) \cong T_x Gx \prec \mathfrak{h}_x.$$

If $\varepsilon = \omega$, then by Theorem 3.2(2), for any $\tilde{x} \in \pi^{-1}(x)$, the homomorphism $\lambda_{\tilde{x}}$ defined in (4.1) restricts to an isomorphism from $\text{Kill}(\tilde{M})$ onto $\text{Kill}_x^{\text{germ}}(M)$. This implies that $\lambda_{\tilde{x}}(\mathcal{W}) = \mathfrak{a}_x$. So

$$d\pi(T_{\tilde{x}} \tilde{G}\tilde{x}) = T_x Gx \subset \text{ev}_x(\mathfrak{a}_x) = \text{ev}_x(\lambda_{\tilde{x}}(\mathcal{W})) = d\pi(\text{ev}_{\tilde{x}}(\mathcal{W})).$$

Thus $T_{\tilde{x}} \tilde{G}\tilde{x} \subset \text{ev}_{\tilde{x}}(\mathcal{W})$. On the other hand, the fact $\lambda_{\tilde{x}}(\mathcal{W}) = \mathfrak{a}_x$ also implies that the Lie algebra $\mathfrak{a}_x^{\mathcal{W}}$ defined in (4.2) is equal to $Z(\mathfrak{a}_x)$. Thus by Lemma 4.4, we have $\mathfrak{h}_x/Z(\mathfrak{a}_x) \prec \mathcal{L}(\overline{\rho(\Gamma)})$. Now from (8.2) and Lemma 2.9 we obtain

$$\text{ad}(\mathfrak{g}/\mathfrak{g}(x)) \prec \text{ad}(\mathfrak{h}_x) \cong \mathfrak{h}_x/Z(\mathfrak{h}_x) \prec \mathfrak{h}_x/Z(\mathfrak{a}_x) \prec \mathcal{L}(\overline{\rho(\Gamma)}).$$

This completes the proof. \square

Remark 8.1. The assertion $\mathfrak{g}(x) \triangleleft \mathfrak{g}$ in Theorem 1.3(1) in fact holds without the geometric structure. More precisely, if an Ad-discompact Lie group G acts smoothly on a smooth manifold M and preserves a finite Borel measure μ on M , then there exists a G -saturated constructible μ -conull subset $M' \subset M$ such that for every $x \in M$ we have $\mathfrak{g}(x) \triangleleft \mathfrak{g}$. This is well-known if we only require M' to be μ -conull (see, e.g., [41, proof of Cor. 3.6]). The topological requirements on M' can be easily obtained by taking [33, Lem. 2.1] into account. This can be also proved directly by using the continuous Gauss map of the geometric structure $\tau : F^1(M) \rightarrow L(\mathfrak{g}, \mathbb{R}^n)$ defined by $\tau(\beta)(X) = \beta^{-1} \left(\frac{d}{dt} \Big|_{t=0} \exp(-tX)x \right)$, where $\beta \in F^1(M)_x$ is viewed as an invertible linear map $\mathbb{R}^n \rightarrow T_x M$.

The following result asserts that if M is smooth and Gromov's representation theorem does not hold, then the local Killing fields on \tilde{M} are highly non-extendable.

Theorem 8.1. *Let M be a connected smooth manifold with a rigid smooth A-structure and a finite smooth measure μ , and let G be a connected noncompact simple Lie group with finite center. Suppose that G acts faithfully, smoothly, and isometrically on M and preserves μ . Let $\rho : \pi_1(M) \rightarrow \text{GL}(\text{Kill}(\tilde{M})^{\text{Kill}(M)})$ be the representation induced by the deck transformations. Then at least one of the following two assertions holds.*

- (1) $\overline{\rho(\pi_1(M))}$ has a Lie subgroup locally isomorphic to G .
- (2) There exists a G -invariant open dense subset U of M , on which G acts locally freely, such that for every $x \in U$, $\text{Kill}_x^{\text{germ}}(M)$ has a subalgebra \mathfrak{g}_x satisfying the following properties.
 - (i) \mathfrak{g}_x centralizes $\text{Kill}(M)$ and $\text{ev}_x(\mathfrak{g}_x) = T_x Gx$.
 - (ii) The map $-\iota_x^{-1} \circ \text{ev}_x : \mathfrak{g}_x \rightarrow \mathfrak{g}$ is a Lie algebra isomorphism.
 - (iii) For any $\tilde{x} \in \pi^{-1}(x)$, every nonzero element in the image of \mathfrak{g}_x under the natural isomorphism $\text{Kill}_x^{\text{germ}}(M) \rightarrow \text{Kill}_{\tilde{x}}^{\text{germ}}(\tilde{M})$ can not be extended to a global Killing field on \tilde{M} .

To prove Theorem 8.1, we need the following well-known result. We sketch the proof for completeness.

Lemma 8.2. *Let M be a connected C^ε manifold, and let G be a connected semisimple Lie group without compact factors. Suppose that G acts faithfully on M by C^ε diffeomorphisms and preserves a finite smooth measure μ on M . Suppose also that either G has a finite center or $\varepsilon = \omega$. Then G acts locally freely on a G -invariant open dense μ -conull subset of M .*

Proof. The proof is similar to that for the simple group case given in [9, 41]. Let

$$U = \{x \in M \mid G(x) \text{ is discrete}\}.$$

Then U is G -invariant and open. Since μ is smooth, it suffices to prove that U is μ -conull. Let G_i ($1 \leq i \leq k$) be the simple factors of G , and let $M_i = M^{G_i}$. By Remark 8.1, there exists a μ -conull set $M' \subset M$ such that for every $x \in M'$, $G(x)_0$ is a normal subgroup of G , hence is the product of some G_i . This implies that $M' \subset U \cup \bigcup_{i=1}^k M_i$. Thus it suffices to show that $\mu(M_i) = 0$. If G has a finite center, this follows from [40, Prop. 4]. If $\varepsilon = \omega$, then M_i is a proper analytic subset of M and hence, by [9, Cor. 3.6], has measure zero. \square

Proof of Theorem 8.1. Let $\mathcal{W} = \text{Kill}(\widetilde{M})^{\text{Kill}(M)}$. For $x \in M$, let \mathcal{G}_x be the image of \mathcal{G} under the injection $\text{Kill}(M) \rightarrow \text{Kill}_x^{\text{germ}}(M)$, and denote

$$\mathfrak{a}_x = \text{Kill}_x^{\text{germ}}(M)^{\text{Kill}(M)}, \quad \mathfrak{h}_x = \text{Hull}_\Gamma^{\mathcal{W}}(\mathfrak{a}_x).$$

By Lemma 1.1(1), G is discompact with respect to the natural representation $G \rightarrow \text{GL}(\text{Kill}(M))$. Similar to the proof of Theorem 1.3, we obtain a G -saturated constructible set $M' \subset M_{\text{reg}}$ with $\mu(M_{\text{reg}} \setminus M') = 0$ such that for every $x \in M'$, there exist $\mathfrak{z}_x < \mathfrak{h}_x$ and a Lie algebra structure on $T_x Gx$ such that $\text{ev}_x(\mathfrak{z}_x) = T_x Gx$, and such that $\iota_x : \mathfrak{g} \rightarrow T_x Gx$ and $-\text{ev}_x : \mathfrak{z}_x \rightarrow T_x Gx$ are Lie algebra homomorphisms. Since μ is smooth, by Corollary 2.2, M' contains a G -invariant open dense subset $U_1 \subset M$. By Lemma 8.2, there exists a G -invariant open dense subset $U_2 \subset M$ such that $\mathfrak{g}(x) = 0$ for every $x \in U_2$. Thus for $x \in U = U_1 \cap U_2$, ι_x is an isomorphism of Lie algebras. In particular, $T_x Gx$ is simple. By using a Levi decomposition of \mathfrak{z}_x , it is easy to see that there exists $\mathfrak{g}_x < \mathfrak{z}_x$ such that $-\text{ev}_x : \mathfrak{g}_x \rightarrow T_x Gx$ is an isomorphism. Obviously, \mathfrak{g}_x satisfies (i) and (ii) in assertion (2).

We prove the theorem by showing that either assertion (1) holds, or for every $x \in U$, \mathfrak{g}_x satisfies (iii) in assertion (2). Note that the latter means that if

$$(8.3) \quad x \in U, \quad \tilde{x} \in \pi^{-1}(x), \quad w \in \text{Kill}(\widetilde{M}), \quad \text{and} \quad \lambda_{\tilde{x}}(w) \in \mathfrak{g}_x,$$

then $w = 0$. For $x \in U$, by Lemma 4.4, there exists a homomorphism $\Phi_x : \mathfrak{h}_x \rightarrow \mathcal{L}(\overline{\rho(\Gamma)})$ such that $\ker(\Phi_x) = \mathfrak{a}_x^{\mathcal{W}}$, where $\mathfrak{a}_x^{\mathcal{W}}$ is as in (4.2). Since $\ker(\Phi_x) \cap \mathfrak{g}_x \triangleleft \mathfrak{g}_x$, it is equal to 0 or \mathfrak{g}_x . If there exists $x \in U$ such that $\ker(\Phi_x) \cap \mathfrak{g}_x = 0$, then

$$\mathfrak{g} \cong \mathfrak{g}_x \cong \Phi_x(\mathfrak{g}_x) < \mathcal{L}(\overline{\rho(\Gamma)}),$$

and hence we get assertion (1). Otherwise, for every $x \in U$, we have $\ker(\Phi_x) \cap \mathfrak{g}_x = \mathfrak{g}_x$, and hence $\mathfrak{g}_x \subset \mathfrak{a}_x^{\mathcal{W}}$. In this case, (8.3) implies that $\lambda_{\tilde{x}}(w) \in \mathfrak{a}_x$, hence centralizes the image of $\text{Kill}(M) \rightarrow \text{Kill}_x^{\text{germ}}(M)$. Since the restriction of $\lambda_{\tilde{x}}$ to $\text{Kill}(\widetilde{M})$ is injective, w centralizes the image of $\text{Kill}(M) \rightarrow \text{Kill}(\widetilde{M})$. So $w \in \mathcal{W}$. This implies that $[\lambda_{\tilde{x}}(w), \mathfrak{g}_x] \subset [\lambda_{\tilde{x}}(w), \mathfrak{a}_x^{\mathcal{W}}] = 0$. Thus $\lambda_{\tilde{x}}(w) \in Z(\mathfrak{g}_x) = 0$, and hence $w = 0$. This completes the proof. \square

A special case of Gromov's open dense orbit theorem is that if G acts topologically transitively on M , then $\text{Iso}^{\text{germ}}(M)$ has an open dense orbit. The following consequence of Theorem 1.2(1) asserts that under certain conditions, a subgroupoid of $\text{Iso}^{\text{germ}}(M)$ already has an open dense orbit.

Theorem 8.3. *Let M be a connected C^∞ manifold with a rigid C^∞ A-structure, G be a connected Lie group which acts on M by C^∞ isometries, and \mathcal{V} be a G -invariant subspace of $\text{Kill}(M)$ such that G is $\rho_{\mathcal{V}}$ -discompact. Suppose that there exists a finite smooth measure μ on M such that $\ker(\rho_{\mathcal{V}}) \cdot G_\mu = G$.*

- (1) *If the G -action is topologically transitive, then $\text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$ has an open dense orbit.*
- (2) *If the G -action is minimal, then $\text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$ is transitive on M .*

Proof. By the $\mathcal{W} = 0$ case of Theorem 1.2(1) and Corollary 2.2, there exists a G -invariant open dense subset U_0 of M such that for every $x \in U_0$, the $\text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$ -orbit of x contains a G -invariant open dense subset of \overline{Gx} . Let $x_0 \in M$ be such that $\overline{Gx_0} = M$. Then any G -invariant open subset of M must contain x_0 . In particular, we have $x_0 \in U_0$. Hence the $\text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$ -orbit O of x_0 contains a G -invariant open dense subset U of M . Obviously, O is dense in M . Thus to prove Theorem 8.3(1), it suffices to show that O is open. Let $x \in O$. Then there exists $\varphi \in \text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$ with source x_0 and target x . Let f be a local isometry defined on an open neighborhood U_f of x_0 such that $\langle f \rangle_{x_0} = \varphi$. Since $x_0 \in U$, we may assume that $U_f \subset U \subset O$. Since φ centralizes \mathcal{V} , we have $\langle df(v|_{U_f}) \rangle_x = \langle v \rangle_x$ for all $v \in \mathcal{V}$. So there exists an open neighborhood $U'_f \subset U_f$ of x_0 such that $df(v|_{U'_f}) = v|_{f(U'_f)}$ for all $v \in \mathcal{V}$. Hence $\langle f \rangle_y \in \text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$ for all $y \in U'_f$. This, together with the fact $U'_f \subset O$, implies that $f(U'_f) \subset O$. Note that $f(U'_f)$ is a neighborhood of x . So O is open. This proves Theorem 8.3(1). If furthermore the G -action is minimal, then as a G -invariant open set, U must be equal to M . Hence $O = M$. Thus $\text{Iso}^{\text{germ}}(M)^{\mathcal{V}}$ is transitive on M . This proves Theorem 8.3(2). \square

Remark 8.2. (1) If $\mathcal{V} = 0$, then G is $\rho_{\mathcal{V}}$ -discompact and the condition $\ker(\rho_{\mathcal{V}}) \cdot G_\mu = G$ is satisfied for any finite smooth measure μ . Thus in this case Theorem 8.3 reduces to the above special case of Gromov's open dense orbit theorem.

- (2) By Lemma 1.1, if μ is G -invariant, then the conditions of Theorem 8.3 are satisfied if G is semisimple without compact factors and $\mathcal{V} = \text{Kill}(M)$, or if $R(G)$ is split solvable, $G/R(G)$ has no compact factors, and $\mathcal{V} = \mathcal{G}$.

9. PROOFS OF THEOREMS 1.4–1.7

In this section we first state two special cases of Theorem 1.3 (Corollaries 9.1 and 9.2 below), and then use them to prove Theorems 1.4–1.7.

Corollary 9.1. *Let M be a connected analytic manifold with a rigid unimodular analytic A-structure of finite volume, let G be a connected Lie subgroup of $\text{Iso}(M)$, and let $\mathcal{V} \supset \mathcal{G}$ be a G -invariant subspace of $\text{Kill}(M)$ such that G is $\rho_{\mathcal{V}}$ -discompact. Then there exist a G -invariant open dense subset $U \subset M$ and a representation $\rho : \pi_1(M) \rightarrow \text{GL}(d, \mathbb{R})$ such that*

- (1) $T_{\tilde{x}} \tilde{G} \tilde{x} \subset \text{ev}_{\tilde{x}}(\text{Kill}(\tilde{M})^{\mathcal{V}})$ for every $\tilde{x} \in \pi^{-1}(U)$, and
- (2) for every $x \in U$, we have $\mathfrak{g}(x) \triangleleft \mathfrak{g}$ and $\text{ad}(\mathfrak{g}/\mathfrak{g}(x)) \prec \mathcal{L} \left(\overline{\rho(\pi_1(M))} \right)$.

Proof. The finite smooth measure μ induced by the unimodular structure is preserved by G . By Theorem 1.3, the conclusions of the corollary hold with U replaced by a G -saturated constructible set $M' \subset M_{\text{reg}}$ with $\mu(M_{\text{reg}} \setminus M') = 0$. Since μ is smooth, by Corollary 2.2, M' contains a G -invariant open dense subset U of M . This completes the proof. \square

Remark 9.1. Let G be the discompact radical of $\text{Iso}(M)$ with respect to the natural representation of $\text{Iso}(M)$ in $\text{Kill}(M)$ (see Proposition 5.6). Then G is a connected closed normal subgroup of $\text{Iso}(M)$ such that $\text{Iso}(M)/G$ is locally isomorphic to a compact Lie group, and the conditions of Corollary 9.1 are satisfied for $\mathcal{V} = \text{Kill}(M)$. This justifies Remark 1.1(2).

Corollary 9.2. *Let M be a connected compact analytic manifold with a rigid analytic A -structure, and let G be a connected split solvable Lie group which acts analytically and isometrically on M . Then for any G -minimal set $M_0 \subset M$ the following assertions hold.*

- (1) *If $\mathcal{V} \supset \mathcal{G}$ is a G -invariant subspace of $\text{Kill}(M)$ such that G is $\rho_{\mathcal{V}}$ -discompact, then for any $\tilde{x} \in \pi^{-1}(M_0)$, we have $T_{\tilde{x}}\tilde{G}\tilde{x} \subset \text{ev}_{\tilde{x}}(\text{Kill}(\tilde{M})^{\mathcal{V}})$. In particular, we always have $T_{\tilde{x}}\tilde{G}\tilde{x} \subset \text{ev}_{\tilde{x}}(\text{Kill}(\tilde{M})^{\mathcal{G}})$ for any $\tilde{x} \in \pi^{-1}(M_0)$.*
- (2) *There exists a representation $\rho : \pi_1(M) \rightarrow \text{GL}(d, \mathbb{R})$ such that for any $x \in M_0$, we have $\mathfrak{g}(x) \triangleleft \mathfrak{g}$ and $\text{ad}(\mathfrak{g}/\mathfrak{g}(x)) \prec \mathcal{L}(\overline{\rho(\pi_1(M))})$.*

Proof. Since G is solvable, it is amenable. So there exists a G -invariant finite Borel measure μ on M supported on M_0 . Since M is analytic and compact, we have $M_{\text{reg}} = M$. By Lemma 1.1, G is $\rho_{\mathcal{G}}$ -discompact. Thus Theorem 1.3 implies that the conclusions of the corollary hold with M_0 replaced by a G -saturated constructible μ -conull subset M' of M . By Lemma 2.7, we have $M_0 \subset M'$, and the corollary follows. \square

Remark 9.2. By [33, Lem. 2.6], $\mathfrak{g}(x)$ is independent of the choice of $x \in M_0$.

Now we prove Theorems 1.4–1.7.

Proof of Theorem 1.4. We prove the theorem by showing that if $\text{Iso}(M)_0$ is non-trivial and has a discrete center, then it is compact and semisimple. Since M is compact, every Killing field on M is complete. So the infinitesimal action $\mathcal{L}(\text{Iso}(M)) \rightarrow \text{Kill}(M)$ induced by the $\text{Iso}(M)$ -action is an $\text{Iso}(M)$ -equivariant isomorphism. Let G be the Ad-discompact radical of $\text{Iso}(M)$. Then G is discompact with respect to its natural representation in $\text{Kill}(M)$, and $\text{Iso}(M)/G$ is locally isomorphic to a compact Lie group. Since M is simply connected, the $\mathcal{V} = \text{Kill}(M)$ case of Corollary 9.1(1) implies that there exists an open dense subset $U \subset M$ such that $T_x Gx \subset \text{ev}_x(Z(\text{Kill}(M)))$ for every $x \in U$. Since $\mathcal{L}(\text{Iso}(M)) \cong \text{Kill}(M)$ and $\text{Iso}(M)_0$ has a discrete center, we have $Z(\text{Kill}(M)) = 0$. Thus for every $x \in U$ we have $T_x Gx = 0$, i.e., x is fixed by G . But U is dense in M . So G acts trivially on M . Hence G is trivial. Thus $\text{Iso}(M)_0$ is locally isomorphic to a compact Lie group. But it has a discrete center. So it is compact and semisimple. \square

Remark 9.3. In the proof of Theorem 1.4, the compactness of M is only used to ensure that every Killing field is complete. If the geometric structure is a pseudo-Riemannian structure or a linear connection plus a volume density, this remains

true if M is not compact but geodesically complete (see [23, Thm. VI.2.4]). This justifies Remark 1.2.

The following lemma will be used in the proofs of Theorems 1.5 and 1.7.

Lemma 9.3. *Let Γ be a group, and let $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{R})$ be a representation. If Γ is virtually abelian (resp. virtually nilpotent), then $\mathcal{L}(\overline{\rho(\Gamma)})$ is abelian (resp. nilpotent).*

Proof. Let Γ_0 be an abelian (resp. nilpotent) subgroup of Γ of finite index. By [8, p. 60, Cor.2], $\mathcal{L}(\overline{\rho(\Gamma_0)})$ is abelian (resp. nilpotent). Let $\gamma_1, \dots, \gamma_k \in \Gamma$ be such that $\Gamma = \bigcup_{i=1}^k \gamma_i \Gamma_0$. Then $\overline{\rho(\Gamma)} = \bigcup_{i=1}^k \rho(\gamma_i) \overline{\rho(\Gamma_0)}$. Thus $\overline{\rho(\Gamma_0)}$ is a closed subgroup of finite index in $\overline{\rho(\Gamma)}$. Hence $\mathcal{L}(\overline{\rho(\Gamma)}) = \mathcal{L}(\overline{\rho(\Gamma_0)})$ is abelian (resp. nilpotent). \square

Proof of Theorem 1.5. We may view G as a connected Lie subgroup of $\mathrm{Iso}(M)$. Since G is split solvable, by Lemma 1.1(1), it is ρ_G -discompact. Then the $\mathcal{V} = \mathcal{G}$ case of Corollary 9.1(2) implies that there exist an open dense subset $U \subset M$ and a representation $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{R})$ such that for every $x \in U$, we have $\mathfrak{g}(x) \triangleleft \mathfrak{g}$ and $\mathrm{ad}(\mathfrak{g}/\mathfrak{g}(x)) \prec \mathcal{L}(\overline{\rho(\Gamma)})$, where $\Gamma = \pi_1(M)$. Note that since U is dense in M and the G -action on M is faithful, we have $\bigcap_{x \in U} \mathfrak{g}(x) = 0$. We prove the theorem by showing that if Γ is finite (resp. virtually abelian, virtually nilpotent), then G is abelian (resp. at most 2-step nilpotent, nilpotent).

(1) If Γ is finite, then $\mathcal{L}(\overline{\rho(\Gamma)}) = 0$. So for every $x \in U$ we have $\mathrm{ad}(\mathfrak{g}/\mathfrak{g}(x)) = 0$, and hence $\mathfrak{g}/\mathfrak{g}(x)$ is abelian. This implies that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}(x)$ for every $x \in U$. So $[\mathfrak{g}, \mathfrak{g}] \subset \bigcap_{x \in U} \mathfrak{g}(x) = 0$. Hence G is abelian.

(2) If Γ is virtually abelian, then by Lemma 9.3, $\mathcal{L}(\overline{\rho(\Gamma)})$ is abelian. So for every $x \in U$, $\mathrm{ad}(\mathfrak{g}/\mathfrak{g}(x))$ is abelian. This implies that $\mathfrak{g}/\mathfrak{g}(x)$ is at most 2-step nilpotent, and hence $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \subset \mathfrak{g}(x)$ for every $x \in U$. Thus $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \subset \bigcap_{x \in U} \mathfrak{g}(x) = 0$. Hence G is at most 2-step nilpotent.

(3) If Γ is virtually nilpotent, then by Lemma 9.3, $\mathcal{L}(\overline{\rho(\Gamma)})$ is nilpotent. So for every $x \in U$, $\mathrm{ad}(\mathfrak{g}/\mathfrak{g}(x))$, hence $\mathfrak{g}/\mathfrak{g}(x)$, is nilpotent. This implies that $\mathfrak{g}_\infty \subset \mathfrak{g}(x)$ for every $x \in U$, where $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \dots$ is the lower central series of \mathfrak{g} and $\mathfrak{g}_\infty = \bigcap_{i=1}^\infty \mathfrak{g}_i$. Thus $\mathfrak{g}_\infty \subset \bigcap_{x \in U} \mathfrak{g}(x) = 0$. Hence G is nilpotent. \square

Proof of Theorem 1.6. Let $M_0 \subset M$ be a G -minimal set. Since M is compact, we have $\mathcal{L}(\mathrm{Iso}(M)) \cong \mathrm{Kill}(M)$, and the adjoint representation of $\mathrm{Iso}(M)$ is equivalent to the natural representation of $\mathrm{Iso}(M)$ in $\mathrm{Kill}(M)$. Thus G is ρ_V -discompact for $\mathcal{V} = \mathrm{Kill}(M)$, and we have $Z(\mathrm{Kill}(M)) = 0$. By Corollary 9.2(1), for every $x \in M_0$ we have $T_x G x \subset \mathrm{ev}_x(Z(\mathrm{Kill}(M))) = 0$. So x is fixed by G . Hence G acts trivially on M_0 . But M_0 is G -minimal. So it consists of a single point. \square

Remark 9.4. If H is a connected noncompact semisimple Lie subgroup of $\mathrm{Iso}(M)$ with an Iwasawa decomposition $H = KAN$, then $G = AN$ satisfies condition (1) in Theorem 1.6. Indeed, if we denote the adjoint representation of $\mathrm{Iso}(M)$ by ρ , then $\rho(H) = \rho(K)\rho(A)\rho(N)$ is an Iwasawa decomposition of $\rho(H)$. Thus $\overline{\rho(G)} = \overline{\rho(A)\rho(N)}$ is an \mathbb{R} -split solvable real algebraic group, and hence is discompact by Corollary 5.2.

Proof of Theorem 1.7. Let $M_0 \subset M$ be a G -minimal set. By Corollary 9.2(2), there exists a representation $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{R})$ such that for every $x \in M_0$, we have $\mathfrak{g}(x) \triangleleft \mathfrak{g}$ and $\mathrm{ad}(\mathfrak{g}/\mathfrak{g}(x)) \prec \mathcal{L}(\overline{\rho(\Gamma)})$, where $\Gamma = \pi_1(M)$. We prove that under either condition of the theorem, we have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}(x)$ for every $x \in M_0$.

(1) Suppose that Γ is finite. Then $\mathcal{L}(\overline{\rho(\Gamma)}) = 0$. This implies that $\mathrm{ad}(\mathfrak{g}/\mathfrak{g}(x)) = 0$ for every $x \in M_0$. So $\mathfrak{g}/\mathfrak{g}(x)$ is abelian, and hence $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}(x)$.

(2) Suppose that Γ is virtually nilpotent and $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ be the lower central series of \mathfrak{g} , and let $\mathfrak{g}_\infty = \bigcap_{i=1}^\infty \mathfrak{g}_i$. Then $\mathfrak{g}_\infty = [\mathfrak{g}, \mathfrak{g}]$. By Lemma 9.3, $\mathcal{L}(\overline{\rho(\Gamma)})$ is nilpotent. So for every $x \in M_0$, $\mathrm{ad}(\mathfrak{g}/\mathfrak{g}(x))$, and hence $\mathfrak{g}/\mathfrak{g}(x)$, is nilpotent. Thus $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_\infty \subset \mathfrak{g}(x)$.

Now let $x \in M_0$. Since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}(x)$, we have $(G, G) \subset G(x)$. This means that x is fixed by (G, G) . So (G, G) acts trivially on M_0 . This completes the proof. \square

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LMAM, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING, 100871, CHINA
E-mail address: anjinpeng@gmail.com